**Exercise 9.1**. a) Write out the three components of (9.1) in *x*-*y*-*z* Cartesian coordinates.
b) Set **u** = (*u*(*y*), 0, 0), and show that the *x*- and *y*-momentum equations reduce to:

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{d^2u}{dy^2} \ , \text{ and } \ 0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} .$$

**Solution 9.1**. a) Equation (9.1) is the constant-viscosity Navier-Stokes' momentum equation for incompressible flow:

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} \ ,$$

where $\nu$ is the kinematic viscosity of the flow. Using **u** = (*u*, *v*, *w*) and $\nabla = \left(\partial/\partial x, \partial/\partial y, \partial/\partial z\right)$, the three components of this equation become:

*x*: $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right),$$

*y*: $$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right), \text{ and}$$

*z*: $$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right).$$

b) When **u** = (*u*(*y*), 0, 0), all the terms involving *v* and *w* disappear, so the part a) equations simplify to:

*x*: $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + 0 + 0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right),$$

*y*: $$0 + 0 + 0 + 0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(0 + 0 + 0\right), \text{ and}$$

*z*: $$0 + 0 + 0 + 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(0 + 0 + 0\right).$$

And, when *u* depends only on *y*, then $\partial u/\partial t = \partial u/\partial x = \partial u/\partial z = 0$ so the part a) equations simplify further:

*x*: $$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial y^2}\right),$$

*y*: $$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \text{ and}$$

*z*: $$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z}.$$

These *x*- and *y*-direction equations match those in the problem statement.

**Exercise 9.2**. For steady pressure driven flow between parallel plates (see Figure 8.3), there are 7 parameters: $u(y)$, $U$, $y$, $h$, $\rho$, $\mu$, and $dp/dx$. Determine a dimensionless scaling law for $u(y)$, and rewrite the flow-field solution (8.5) in dimensionless form.

**Solution 9.2**. The parameters are: $u(y)$, $U$, $y$, $h$, $\rho$, $\mu$, and $dp/dx$. First, create the parameter matrix:

| | $u$ | $U$ | $y$ | $h$ | $\rho$ | $\mu$ | $dp/dx$ |
|---|---|---|---|---|---|---|---|
| Mass: | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| Length: | 1 | 1 | 1 | 1 | -3 | -1 | -2 |
| Time: | -1 | -1 | 0 | 0 | 0 | -1 | -2 |

Next, determine the number of dimensionless groups. This rank of this matrix is three so 7 parameters - 3 dimensions = 4 groups, and construct the groups: $\Pi_1 = u/U$, $\Pi_2 = y/h$, $\Pi_3 = \rho U h/\mu$, and $\Pi_4 = h^2(dp/dx)/U\mu$. Now write a dimensionless law:

$$\frac{u}{U} = f\left(\frac{y}{h}, \frac{\rho U h}{\mu}, \frac{h^2}{U\mu}\frac{dp}{dx}\right)$$

where $f$ is an unknown function.

When rewritten in dimensionless form, (8.5) is:

$$\frac{u}{U} = \frac{y}{h} - \frac{h^2}{2U\mu}\frac{dp}{dx}\left(\frac{y}{h}\right)\left(1-\frac{y}{h}\right) \quad \text{or} \quad \Pi_1 = \Pi_2 - \frac{1}{2}\Pi_4\Pi_2(1-\Pi_2).$$

In this flow, there is no fluid acceleration so the Reynolds number, $\Pi_3$, does not appear.

**Exercise 9.3**. An incompressible viscous liquid with density $\rho$ fills the gap between two large smooth parallel walls that are both stationary. The upper and lower walls are located at $x_2 = \pm h$, respectively. An additive in the liquid causes its viscosity to vary in the $x_2$ direction. Here the flow is driven by a constant non-zero pressure gradient: $\partial p/\partial x_1 = const.$

a) Assume steady flow, ignore the body force, set $\mathbf{u} = \left(u_1(x_2),0,0\right)$ and use

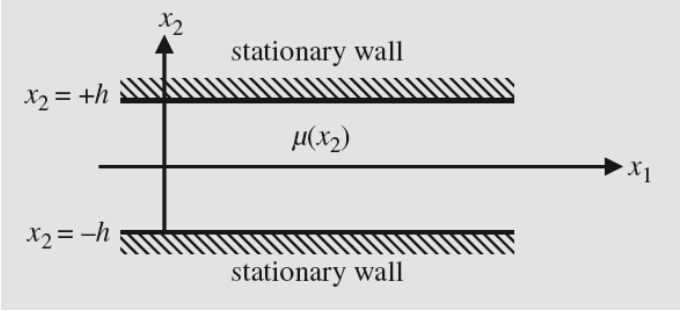
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho u_i\right) = 0 \ , \quad \rho\frac{\partial u_j}{\partial t} + \rho u_i\frac{\partial u_j}{\partial x_i} = -\frac{\partial p}{\partial x_j} + \rho g_j + \frac{\partial}{\partial x_i}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] + \frac{\partial}{\partial x_j}\left[\left(\mu_v - \frac{2}{3}\mu\right)\frac{\partial u_i}{\partial x_i}\right]$$

to determine $u_1(x_2)$ when $\mu = \mu_o\left(1+\gamma\left(x_2/h\right)^2\right)$.

b) What shear stress is felt on the lower wall?

c) What is the volume flow rate (per unit depth into the page) in the gap when $\gamma = 0$?

d) If $-1 < \gamma < 0$, will the volume flux be higher or lower than the case when $\gamma = 0$?




**Solution 9.3**. a) The continuity equation is satisfied by the form of the velocity field. The $j$ =1-component of momentum equation simplifies to:

$$0 = -\left(\partial p/\partial x_1\right) + \left(\partial/\partial x_2\right)\left[\mu\left(\partial u_1/\partial x_2\right)\right].$$

Integrate once with $\partial p/\partial x_1 = const.$ to find: $\mu\left(\partial u_1/\partial x_2\right) = \left(\partial p/\partial x_1\right)x_2 + C$. Divide by $\mu$ and integrate again:

$$u_1 = \int \frac{1}{\mu}\left(\frac{\partial p}{\partial x_1}x_2 + C\right)dx_2 = \int \frac{\left(\partial p/\partial x_1\right)x_2 + C}{\mu_o\left(1+\gamma\left(x_2/h\right)^2\right)}dx_2$$

$$= \frac{h^2}{2\gamma\mu_o}\frac{\partial p}{\partial x_1}\ln\left(1+\gamma\left(\frac{x_2}{h}\right)^2\right) + \frac{Ch}{\mu_o\sqrt{\gamma}}\tan^{-1}\left(\frac{x_2\sqrt{\gamma}}{h}\right) + D.$$

The boundary conditions, $u_1(\pm h) = 0$, determine the values of the constants: $C$ = 0, and $D = -\left(h^2/2\gamma\mu_o\right)\left(\partial p/\partial x_1\right)\ln\left(1+\gamma\right)$, thus:

$$u_1(x_2) = -\frac{h^2}{2\gamma\mu_o}\frac{\partial p}{\partial x_1}\ln\left(\frac{1+\gamma\left(x_2/h\right)^2}{1+\gamma}\right).$$

b) From the solution of part a) with $C = 0$: $\tau_w = \left(\mu\,\partial u_1/\partial x_2\right)_{y=-h} = -h\left(\partial p/\partial x_1\right)$

c) When $\gamma = 0$, the flow profile is parabolic: $q = \int_{-h}^{+h} u_1(x_2)dx_2 = -\frac{h^2}{2\mu_o}\frac{\partial p}{\partial x_1}\int_{-h}^{+h}\left(1-\frac{x_2^2}{h^2}\right)dx_2 = -\frac{2h^3}{3\mu_o}\frac{\partial p}{\partial x_1}$

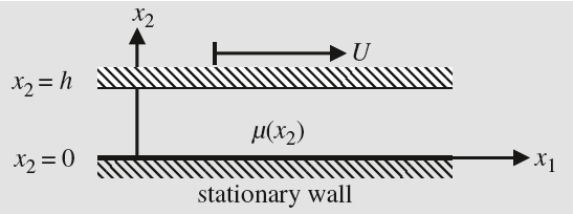
d) The volume flux will be higher because the viscosity will be reduced at the wall. Manipulation of the near-wall viscosity with additives is sometimes used in long piping systems to reduce pumping power requirements.

**Exercise 9.4**. An incompressible viscous liquid with density $\rho$ fills the gap between two large smooth parallel plates. The upper plate at $x_2 = h$ moves in the positive $x_1$-direction at speed $U$. The lower plate at $x_2 = 0$ is stationary. An additive in the liquid causes its viscosity to vary in the $x_2$ direction.

a) Assume steady flow, ignore the body force, set $\mathbf{u} = \left(u_1(x_2),0,0\right)$ and $\partial p/\partial x_1 = 0$, and use the equations specified in Exercise 8.3 to determine $u_1(x_2)$ when $\mu = \mu_o\left(1+\gamma\, x_2/h\right)$.

b) What shear stress is felt on the lower plate?

c) Are there any physical limits on $\gamma$? If, so specify them.




**Solution 9.4**. a) For $\mathbf{u} = \left(u_1(x_2),0,0\right)$, no body force, and $\partial p/\partial x_1 = 0$ in steady incompressible flow, the continuity equation is automatically satisfied, and the momentum equation for $j = 1$ simplifies to:

$$0 = +\left(\partial/\partial x_2\right)\left[\mu\left(\partial u_1/\partial x_2\right)\right], \text{ or, after integrating once: } C = \mu\left(\partial u_1/\partial x_2\right),$$

where $C$ is a constant. Now use the specified relationship for the viscosity and integrate to find:

$$u_1(x_2) = \int \frac{C}{\mu}\,dx_2 = \int \frac{C\,dx_2}{\mu_o\left(1+\gamma\, x_2/h\right)} = \frac{Ch}{\mu_o\gamma}\ln\left(1+\gamma\, x_2/h\right) + D$$

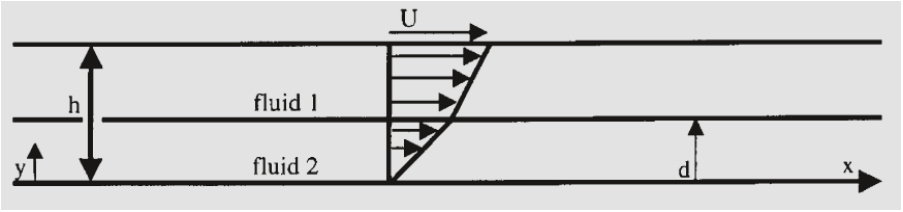
where $D$ is another constant. The boundary conditions $u_1(0) = 0$ and $u_1(h) = U$ allow $C = U\gamma\mu_o/\left(h\ln(1+\gamma)\right)$ and $D = 0$ to be determined yielding:

$$u_1(x_2) = U\ln\left(1+\gamma\, x_2/h\right)/\ln\left(1+\gamma\right).$$

b) From part a), the shear stress is constant: $\tau_w = \mu\left(\partial u_1/\partial x_2\right) = C = U\gamma\mu_o/\left(h\ln(1+\gamma)\right)$.

c) Negative viscosities violate the second law of thermodynamics, thus $\gamma > -1$ is required.

**Exercise 9.5**. Planar Couette flow is generated by placing a viscous fluid between two infinite parallel plates and moving one plate (say, the upper one) at a velocity $U$ with respect to the other one. The plates are a distance $h$ apart. Two immiscible viscous liquids are placed between the plates as shown in the diagram. The lower fluid layer has thickness $d$. Solve for the velocity distributions in the two fluids.




**Solution 9.5**. For steady viscous flow between infinite parallel plates, the fluid velocity will be unidirectional: $\mathbf{u} = (u, 0, 0)$. For this problem, no pressure gradient is specified so assume it to be zero. Thus, the horizontal ($x_1$-direction) momentum equation reduces to:

$$\partial\tau_{21}/\partial x_2 = \left(\partial/\partial x_2\right)\left(\mu\,\partial u/\partial x_2\right) = \mu\partial^2 u/\partial y^2 = 0\,,$$

where the last equality follows when the viscosity is constant and $x_2 = y$. Here, the viscosity is assumed constant within each fluid. This means that the flow profile in each fluid will be piece-wise linear:

$$u(y) = \begin{cases} A_1 + B_1 y \;\text{ for }\; 0 \le y \le d \\ A_2 + B_2 y \;\text{ for }\; d \le y \le h \end{cases},$$

where the $A$'s & $B$'s are constants and '1' implies the upper fluid layer with viscosity $\mu_1$, and '2' implies the lower fluid layer with viscosity $\mu_2$. The four constants can be determined from the four boundary conditions:

| | |
|---|---|
| *i*) $u(0) = 0$ | (match the speed of the lower boundary) |
| *ii*) $u(h) = U$ | (match the speed of the upper boundary) |
| *iii*) $u(d^-) = u(d^+)$, and | (match flow speeds at the internal fluid-fluid interface) |
| iv) $\tau(d^-) = \tau(d^+)$ | (match shear stress at the internal fluid-fluid interface) |

where $\tau$ is the shear stress in the fluid. These four boundary conditions imply:

$$A_2 = 0,\; A_1 + B_1 h = U\,,\; A_2 + B_2 d = A_1 + B_1 d\,,\text{ and }\; \mu_2 B_2 = \mu_1 B_1$$

Use the first two equations to eliminate $A_1$ and $A_2$ from the second two equations to find:

$$B_2 d = U - B_1(h - d)\,,\text{ and }\; \mu_2 B_2 = \mu_1 B_1.$$

Eliminate $B_2$ and solve for $B_1$:

$$\left(\mu_1/\mu_2\right)B_1 d = U - B_1(h - d) \;\text{ –> }\; B_1 = \mu_2 U/\left[\mu_2 h + \left(\mu_1 - \mu_2\right)d\right].$$

So, $B_2 = \mu_1 U/\left[\mu_2 h + \left(\mu_1 - \mu_2\right)d\right]$, and $A_1 = U\left(\mu_1 - \mu_2\right)d/\left[\mu_2 h + \left(\mu_1 - \mu_2\right)d\right]$ with $A_2 = 0$. Thus:

$$u(y) = \frac{U}{\mu_2 h + \left(\mu_1 - \mu_2\right)d}\begin{cases} \mu_1 y \;\text{ for }\; 0 \le y \le d \\ \left(\mu_1 - \mu_2\right)d + \mu_2 y \;\text{ for }\; d \le y \le h \end{cases}$$

**Exercise 9.6**. Consider plane Poiseuille flow of a non-Newtonian power-law fluid using the coordinate system and geometry shown in Figure 9.3. Here the fluid's constitutive relationship is given by (4.37): $\tau_{xy} = m\left(\partial u/\partial y\right)^n$ , where $m$ is the power law coefficient, and $n$ is the power law exponent.
a) Determine the velocity profile $u(y)$ in the lower half of the channel, $0 \le y \le h/2$, using the boundary condition $u(0) = 0$.
b) Given that the maximum velocity occurs at $y = h/2$ and that the flow profile is symmetric about this location, plot $u(y)/u(h/2)$ vs. $y/(h/2)$ for $n = 2$ (a shear thickening fluid), $n = 1$ (a Newtonian fluid), and $n = 0.4$ (a shear thinning fluid).
c) Explain in physical terms why the shear-thinning velocity profile is the bluntest.

**Solution 9.6**. In plane Poiseuille flow, the upper channel wall does not move. In the lower half of the channel, the $\partial u/\partial y > 0$, so fractional powers are readily managed.
a) For fully developed constant density flow (so that $\partial p/\partial x$ is independent of $x$) the horizontal momentum equation reduces to:

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} .$$

The vertical momentum equation is simply, $\partial p/\partial y = 0$, so $\partial p/\partial x$ does not depend on $x$ or $y$. Thus, this horizontal momentum equation can be integrated to find:

$$y\frac{\partial p}{\partial x} = \tau_{xy} + C ,$$

where $C$ is a constant. However, when $y = h/2$, $\tau_{xy} = 0$ so $C = (h/2)\partial p/\partial x$, therefore:

$$\left(\frac{h}{2} - y\right)\left(-\frac{\partial p}{\partial x}\right) = \tau_{xy} = m\left(\frac{\partial u}{\partial y}\right)^n , \text{ or } \frac{\partial u}{\partial y} = \left(-\frac{1}{m}\frac{\partial p}{\partial x}\right)^{1/n}\left(\frac{h}{2} - y\right)^{1/n} ,$$

and both equations are written this way because $-\partial p/\partial x$ is positive. The second equation can be integrated to find:

$$u(y) = -\left(\frac{1}{m}\frac{\partial p}{\partial x}\right)^{1/n}\frac{1}{1+1/n}\left(\frac{h}{2} - y\right)^{1+1/n} + D ,$$

where $D$ is another constant of integration. The horizontal velocity must go to zero on the lower channel wall at $y = 0$, so

$$D = \left(\frac{1}{m}\frac{\partial p}{\partial x}\right)^{1/n}\frac{1}{1+1/n}\left(\frac{h}{2}\right)^{1+1/n} .$$

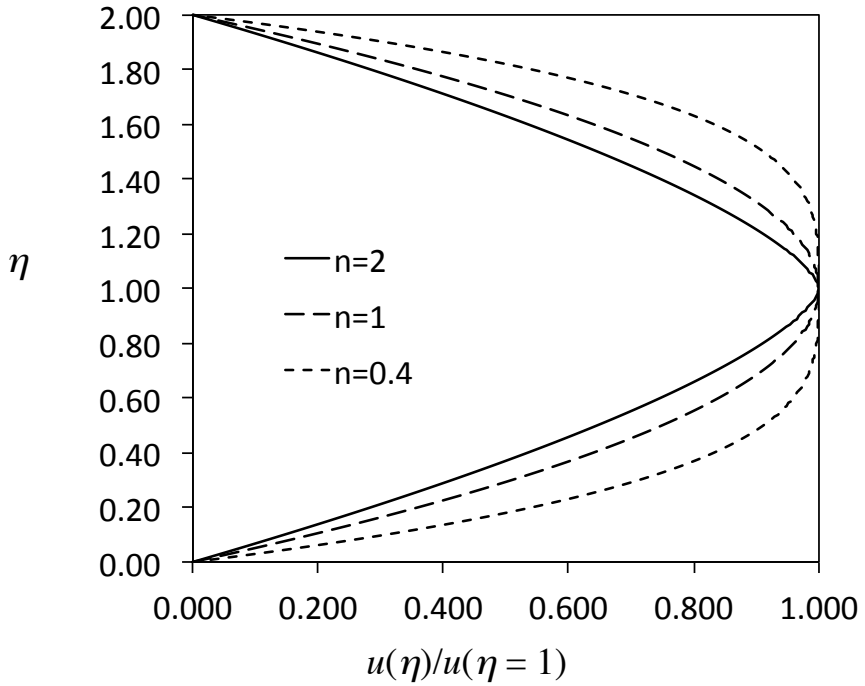
Thus, the velocity profile is:

$$u(y) = \frac{n}{n+1}\left(\frac{1}{m}\frac{\partial p}{\partial x}\right)^{1/n}\left[\left(\frac{h}{2}\right)^{1+1/n} - \left(\frac{h}{2} - y\right)^{1+1/n}\right] = \left(\frac{n}{n+1}\right)\frac{h}{2}\left(\frac{h}{2m}\frac{\partial p}{\partial x}\right)^{1/n}\left[1 - \left(1 - \frac{y}{h/2}\right)^{1+1/n}\right] .$$

b) Evaluate the result of part a) at $y = h/2$, and form the ratio:

$$\frac{u(y)}{u(h/2)} = \frac{\left(\frac{n}{n+1}\right)\frac{h}{2}\left(\frac{h}{2m}\frac{\partial p}{\partial x}\right)^{1/n}\left[1 - \left(1 - \frac{y}{h/2}\right)^{1+1/n}\right]}{\left(\frac{n}{n+1}\right)\frac{h}{2}\left(\frac{h}{2m}\frac{\partial p}{\partial x}\right)^{1/n}} = \left[1 - \left(1 - \eta\right)^{1+1/n}\right] ,$$

where $\eta = y/(h/2)$. The three curves for $n = 2$, 1, and 0.4 are plotted here:

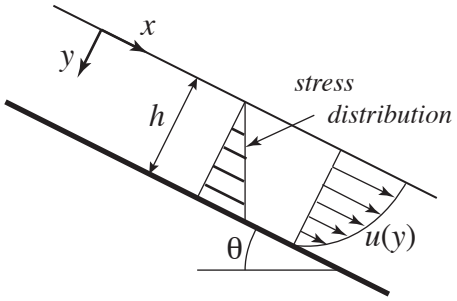



c) From the part a) solution, the shear stress is:

$$\tau_{xy} = -\frac{\partial p}{\partial x}\left(\frac{h}{2} - y\right).$$

The highest magnitude shear stress occurs at the upper and lower wall of the channel. In a shear thinning fluid, the local viscosity decreases when the shear stress increases. Thus, to maintain this shear stress profile, the velocity gradient magnitude must increase in regions where the shear stress magnitude is also large. This leads to a blunter profile for a shear thinning fluid compared to a Newtonian (or a shear thickening) fluid.

**Exercise 9.7**. Consider the laminar flow of a fluid layer falling down a plane inclined at an angle $\theta$ with respect to the horizontal. If $h$ is the thickness of the layer in the fully developed stage, show that the velocity distribution is $u(y) = \left(g/2\nu\right)\left(h^2 - y^2\right)\sin\theta$, where the $x$-axis points in the direction of flow along the free surface, and the $y$-axis points toward the plane. Show that the volume flow rate per unit width is $Q = \left(gh^3/3\nu\right)\sin\theta$, and that and the frictional stress on the wall is $\tau_w = \rho g h \sin\theta$.




**Solution 9.7**. In the given coordinates, there is a component of gravity, $g\sin\theta$, acting in the $x$-direction, but the presence of atmospheric pressure on the liquid surface ensures that $\partial p/\partial x = 0$. Thus, the steady flow $x$-direction momentum equation is:

$$0 = +g\sin\theta + \nu\frac{\partial^2 u}{\partial y^2}.$$

Integrating twice produces:

$$u(y) = -\frac{g\sin\theta}{2\nu}y^2 + Ay + B,$$

where $A$ and $B$ are constants. The boundary condition on the liquid surface ($y = 0$) is zero shear stress ($\mu\partial u/\partial y = 0$), and this allows $A$ to be evaluated:

$$0 = \left[\frac{\partial u}{\partial y}\right]_{y=0} = \left[-\frac{g\sin\theta}{\nu}y + A\right]_{y=0} = A.$$

The boundary condition, on the solid surface ($y = h$) is zero velocity ($u = 0$), and this allows $B$ to be evaluated:

$$0 = u(h) = -\frac{g\sin\theta}{2\nu}h^2 + B, \text{ or } B = \frac{g\sin\theta}{2\nu}h^2.$$

so the velocity profile is:

$$u(y) = -\frac{g\sin\theta}{2\nu}\left(y^2 - h^2\right).$$

The volume flow rate per unit width is:

$$Q = \int_0^h u(y)dy = -\frac{g\sin\theta}{2\nu}\int_0^h\left(y^2 - h^2\right)dy = -\frac{g\sin\theta}{2\nu}\left(\frac{h^3}{3} - h^3\right) = \frac{g\sin\theta}{3\nu}h^3.$$

The magnitude of the shear stress is:

$$\tau = \mu\frac{\partial u}{\partial y} = \mu\left(\frac{g\sin\theta}{\nu}\right)y = \rho g y\sin\theta,$$

and the maximum shear stress, $\tau_w = \rho g h \sin\theta$, occurs at the solid surface.

**Exercise 9.8**. In two-dimensional $(x,y)$-coordinates, the Navier-Stokes equations for the fluid velocity, $\mathbf{u} = (u,v)$, in a constant-viscosity constant-density flow are: $\partial u/\partial x + \partial v/\partial y = 0$,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad \text{and} \quad \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right).$$

a) Cross differentiate and sum the two momentum equations to reach the following equation for $\omega_z = \partial v/\partial x - \partial u/\partial y$, the vorticity normal to the $x$-$y$ plane:

$$\frac{\partial \omega_z}{\partial t} + u\frac{\partial \omega_z}{\partial x} + v\frac{\partial \omega_z}{\partial y} = \nu\left(\frac{\partial^2 \omega_z}{\partial x^2} + \frac{\partial^2 \omega_z}{\partial y^2}\right).$$

b) The simplest nontrivial solution of this equation is uniform shear or solid body rotation ($\omega_z$ = constant). The next simplest solution is a linear function of the independent coordinates: $\omega_z = ax + by$, where $a$ and $b$ are constants. Starting from this vorticity field, derive the following velocity field:

$$u = -\frac{b}{2}\left[\frac{(ax+by)^2}{a^2+b^2} + c\right] \quad \text{and} \quad v = \frac{a}{2}\left[\frac{(ax+by)^2}{a^2+b^2} + c\right].$$

where $c$ is an undetermined constant.

c) For the part b) flow, sketch the streamlines. State any assumptions you make about $a$, $b$, and $c$.

d) For the part b) flow when $a = 0$, $b > 0$, and $\mathbf{u} = (U_o, 0)$ at the origin of coordinates with $U_o > 0$, sketch the velocity profile along a line $x$ = constant, and determine $\nabla p$.

**Solution 9.8**. a) Apply $-\partial/\partial y$ to the $x$-direction momentum equation, and $\partial/\partial x$ to the $y$-direction momentum equation to reach:

$$-\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right) - \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} - u\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) - \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} - v\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{1}{\rho}\frac{\partial^2 p}{\partial y \partial x} - \nu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial u}{\partial y}\right), \text{ and}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + v\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x \partial y} + \nu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial v}{\partial x}\right).$$

Add these two equations together noting that pressure terms cancel, and that the second and fourth terms in each equation sum to zero because of the continuity equation, $\partial u/\partial x + \partial v/\partial y = 0$.

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \nu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

Substituting in the definition $\omega_z = \partial v/\partial x - \partial u/\partial y$ leads to the final form:

$$\frac{\partial \omega_z}{\partial t} + u\frac{\partial \omega_z}{\partial x} + v\frac{\partial \omega_z}{\partial y} = \nu\left(\frac{\partial^2 \omega_z}{\partial x^2} + \frac{\partial^2 \omega_z}{\partial y^2}\right).$$

b) To find the velocity field start with: $\omega_z = ax + by = \partial v/\partial x - \partial u/\partial y$. (1)

The part a) equation implies: $ua + vb = 0$ or $v = -(a/b)u$. (2)

The continuity equation is also needed: $\partial u/\partial x + \partial v/\partial y = 0$. (3)

Use (2) to eliminate $v$, from (1) and (3) to find:

$$ax + by = -\frac{a}{b}\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}, \quad \text{and} \quad \frac{\partial u}{\partial x} - \frac{a}{b}\frac{\partial u}{\partial y} = 0.$$

Combine these twice, first to eliminate $\partial u/\partial y$, then to eliminate $\partial u/\partial x$ to reach:

$$ax+by=-\left(\frac{a^2}{b^2}+1\right)\frac{\partial u}{\partial y},\ \text{and}\ \ ax+by=-\left(\frac{a}{b}+\frac{b}{a}\right)\frac{\partial u}{\partial x}.$$

Integrate to find: $axy+b\frac{y^2}{2}=-\left(\frac{a^2}{b^2}+1\right)u+f(x)$, and $a\frac{x^2}{2}+bxy=-\left(\frac{a}{b}+\frac{b}{a}\right)u+g(y)$, where $f(x)$ and $g(y)$ are unknown functions. Solve both equations for $u$ and absorb multiplicative constants into $f(x)$ and $g(y)$.

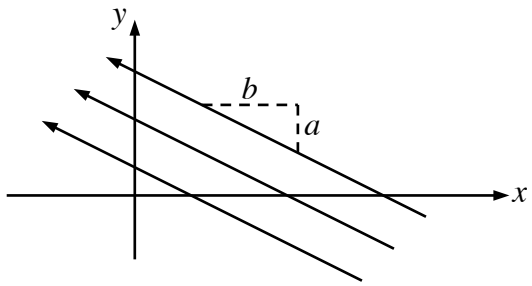
$$u=-\frac{b}{2}\left(\frac{f(x)+2abxy+b^2y^2}{a^2+b^2}\right),\ \text{and}\ \ u=-\frac{b}{2}\left(\frac{a^2x^2+ab^2xy+g(y)}{a^2+b^2}\right).$$

For consistency, the unknown functions must be: $f(x) = a^2x^2 + const.$ and $g(y) = b^2y^2 + const.$ so

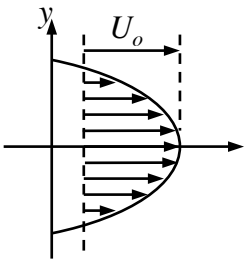
$$u=-\frac{b}{2}\left(\frac{(ax+by)^2}{a^2+b^2}+c\right),\ \text{and then from (2)}$$

$$v=-\frac{a}{b}u=\frac{a}{2}\left(\frac{(ax+by)^2}{a^2+b^2}+c\right).$$

where $c$ is another undetermined constant.




c) $dy_{sl}/dx_{sl} = v/u = -a/b$. Thus, if $a$ and $b$ are both positive constants, the streamlines are negatively-slopped straight lines. In this case, $u < 0$ and $v > 0$, so the arrows point upward and to the left.




d) Evaluate the constants to find: $u=U_o-(b/2)y^2$ and $v = 0$, so the horizontal velocity profile is parabolic. To get the pressure gradient, evaluate the 2D Navier-Stokes equations using the part b) velocity field:

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu(-b),\ \text{and}\ 0=-\frac{1}{\rho}\frac{\partial p}{\partial y}+0,\ \text{so}\ \ \vec{\nabla}p=(-\mu b,0).$$

**Exercise 9.9**. Consider circular Couette flow as described by (9.10) in the limit of a thin gap between the cylinders. Use the definitions: $\overline{R} = (R_1 + R_2)/2$, $h = R_2 - R_1$, $\overline{\Omega} = (\Omega_1 + \Omega_2)/2$, $\Delta\Omega = \Omega_2 - \Omega_1$, and $R = \overline{R} + y$ to complete the following items.

a) Show that $u_\varphi(y) \cong \overline{\Omega}\overline{R} + \Delta\Omega\overline{R}\left(y/h\right)$ when $y << \overline{R}$, $h << \overline{R}$, and all terms of order $y/\overline{R}$ and $h/\overline{R}$ or higher have been dropped. This is the lubrication approximation for circular Couette flow.

b) Compute the shear stress $\mu\left(du_\varphi/dy\right)$ from the part a) flow field and put it terms of $\mu$, $R_1$, $R_2$, $\Omega_1$, and $\Omega_2$.

c) Compute the exact shear stress from (9.10) using: $\tau_{R\varphi} = 2\mu\left[\frac{R}{2}\frac{\partial}{\partial R}\left(\frac{u_\varphi}{R}\right) + \frac{1}{2R}\frac{\partial u_R}{\partial \varphi}\right]$, and evaluate it at $R = R_1$, and $R = R_2$.

d) What values of $h/\overline{R}$ lead to shear stress errors of 1%, 3% and 10%.

**Solution 9.9**. Here, $R_1 = \overline{R} - h/2$ and $R_2 = \overline{R} + h/2$, so $R_1^2 = \overline{R}^2 - \overline{R}h + h^2/4$ and $R_2^2 = \overline{R}^2 + \overline{R}h + h^2/4$. Plus, $R_2^2 - R_1^2 = 2\overline{R}h$. Use these exact relationships in (9.10) to reach:

$$u_\varphi(y) = \frac{1}{2\overline{R}h}\left\{\left[\Omega_2\left(\overline{R}^2 + \overline{R}h + \frac{h^2}{4}\right) - \Omega_1\left(\overline{R}^2 - \overline{R}h + \frac{h^2}{4}\right)\right]\left(\overline{R} + y\right) - \Delta\Omega\frac{\left(\overline{R}^2 - \overline{R}h + \frac{h^2}{4}\right)\left(\overline{R}^2 + \overline{R}h + \frac{h^2}{4}\right)}{\overline{R} + y}\right\}.$$

Simplify terms within the large {,}-brackets.

$$u_\varphi(y) = \frac{1}{2\overline{R}h\left(\overline{R} + y\right)}\left\{\left[\Delta\Omega\overline{R}^2 + 2\overline{\Omega}\overline{R}h + \Delta\Omega\frac{h^2}{4}\right]\left(\overline{R} + y\right)^2 - \Delta\Omega\left(\overline{R}^4 - \frac{\overline{R}^2h^2}{2} + \frac{h^4}{16}\right)\right\}.$$

Perform all the multiplications within the large {,}-brackets and continue with this simplification:

$$u_\varphi(y) = \frac{1}{2\overline{R}h\left(\overline{R} + y\right)}\left\{\begin{array}{l}\Delta\Omega\overline{R}^4 + \Delta\Omega 2\overline{R}^3 y + \Delta\Omega\overline{R}^2 y^2 + 2\overline{\Omega}\overline{R}^3 h + 4\overline{\Omega}\overline{R}^2 hy + 2\overline{\Omega}\overline{R}hy^2 + \\ \Delta\Omega\frac{h^2}{4}\overline{R}^2 + \Delta\Omega\frac{h^2}{2}\overline{R}y + \Delta\Omega\frac{h^2}{4}y^2 - \Delta\Omega\overline{R}^4 + \Delta\Omega\frac{\overline{R}^2h^2}{2} - \Delta\Omega\frac{h^4}{16}\end{array}\right\}.$$

Cancel equal and opposite terms, and group terms than involve the average, $\overline{\Omega} = (\Omega_1 + \Omega_2)/2$, and difference, $\Delta\Omega = \Omega_2 - \Omega_1$, rotation rates together.

$$u_\varphi(y) = \frac{1}{2\overline{R}h\left(\overline{R} + y\right)}\left\{2\overline{\Omega}\overline{R}h\left(\overline{R} + y\right)^2 + \Delta\Omega\left(2\overline{R}^3 y + \overline{R}^2\left(\frac{3h^2}{4} + y^2\right) + \overline{R}\frac{h^2}{2}y + \frac{h^2y^2}{4} - \frac{h^4}{16}\right)\right\}.$$

Remove the large {,}-brackets.

$$u_\varphi(y) = \overline{\Omega}\left(\overline{R} + y\right) + \frac{\Delta\Omega\overline{R}}{1 + y/\overline{R}}\left[\frac{y}{h} + \frac{1}{2\overline{R}h}\left(\frac{3h^2}{4} + y^2\right) + \frac{hy}{4\overline{R}^2} + \frac{1}{2\overline{R}^3}\left(\frac{hy^2}{4} - \frac{h^3}{16}\right)\right].$$

Expand $\left(1 + y/\overline{R}\right)^{-1}$ in a power series for $y << \overline{R}$:

$$u_\varphi(y) = \overline{\Omega}\left(\overline{R}+y\right) + \Delta\Omega\overline{R}\left(1-\frac{y}{\overline{R}}+\frac{y^2}{\overline{R}^2}+\ldots\right)\left[\frac{y}{h}+\frac{1}{2\overline{R}h}\left(\frac{3h^2}{4}+y^2\right)+\frac{hy}{4\overline{R}^2}+\frac{1}{2\overline{R}^3}\left(\frac{hy^2}{4}-\frac{h^3}{16}\right)\right] .$$

Keeping only linear terms in $y/\overline{R}$ and $h/\overline{R}$ leads to the final series:

$$u_\varphi(y) = \overline{\Omega}\overline{R} + \Delta\Omega\overline{R}\left(\frac{y}{h}\right) + \overline{\Omega}\overline{R}\left(\frac{y}{\overline{R}}\right) + \ldots \cong \overline{\Omega}\overline{R} + \Delta\Omega\overline{R}\left(\frac{y}{h}\right) ,$$

where the final approximate equality follows because $h << \overline{R}$.

b) $\tau = \mu\dfrac{du_\varphi}{dy} = \dfrac{\Delta\Omega\overline{R}}{h} = \mu\dfrac{(\Omega_2-\Omega_1)(R_2+R_1)}{2(R_2-R_1)}$

c) Here $u_R = 0$, and $\dfrac{u_\varphi(R)}{R} = \dfrac{1}{R_2^2-R_1^2}\left\{\Omega_2R_2^2-\Omega_1R_1^2-\left[\Omega_2-\Omega_1\right]\dfrac{R_1^2R_2^2}{R^2}\right\}$. Thus,

$\dfrac{\partial}{\partial R}\left(\dfrac{u_\varphi(R)}{R}\right) = \dfrac{1}{R_2^2-R_1^2}\left\{+2\left[\Omega_2-\Omega_1\right]\dfrac{R_1^2R_2^2}{R^3}\right\}$ so $\tau_{R\varphi} = \mu R\dfrac{\partial}{\partial R}\left(\dfrac{u_\varphi(R)}{R}\right) = \dfrac{\mu}{R_2^2-R_1^2}\left\{2\left[\Omega_2-\Omega_1\right]\dfrac{R_1^2R_2^2}{R^2}\right\}$.

Evaluating at $R = R_1$, and $R = R_2$ leads to

$$\tau_{R\varphi}(R_1) = \frac{2\mu\left[\Omega_2-\Omega_1\right]}{R_2^2-R_1^2}R_2^2 = \frac{\mu\left[\Omega_2-\Omega_1\right]}{\overline{R}h}\left(\overline{R}+\frac{h}{2}\right)^2 \cong \frac{\mu\left[\Omega_2-\Omega_1\right]\overline{R}}{h}\left(1+\frac{h}{\overline{R}}\right) \quad \text{and}$$

$$\tau_{R\varphi}(R_2) = \frac{2\mu\left[\Omega_2-\Omega_1\right]}{R_2^2-R_1^2}R_1^2 = \frac{\mu\left[\Omega_2-\Omega_1\right]}{\overline{R}h}\left(\overline{R}-\frac{h}{2}\right)^2 \cong \frac{\mu\left[\Omega_2-\Omega_1\right]\overline{R}}{h}\left(1-\frac{h}{\overline{R}}\right) .$$

To first order, the average shear stress is:

$$\overline{\tau} = \frac{1}{2}\left(\tau_{R\varphi}(R_1)+\tau_{R\varphi}(R_2)\right) = \frac{2\mu\left[\Omega_2-\Omega_1\right]}{R_2^2-R_1^2}R_2^2 = \frac{\mu\left[\Omega_2-\Omega_1\right]}{h}\overline{R} ,$$

which matches the result from the lubrication approximation.

d) To first order, the difference of either exact shear stress from this average shear stress is:

$$\frac{\Delta\tau}{\overline{\tau}} = \frac{\tau_{R\varphi}(R_{1or2})-\overline{\tau}}{\overline{\tau}} = \pm\frac{h}{\overline{R}} .$$

Thus, shear stress errors of shear stress errors of 1%, 3% and 10% occur when $h/\overline{R} = 0.01, 0.03,$ and $0.10$.

**Exercise 9.10**. Room temperature water drains through a round vertical tube with diameter $d$. The length of the tube is $L$. The pressure at the tube's inlet and outlet is atmospheric, the flow is steady, and $L >> d$.
a) Using dimensional analysis write a physical law for the mass flow rate $\dot{m}$ through the tube.
b) Assume that the velocity profile in the tube is independent of the vertical coordinate, determine a formula for $\dot{m}$, and put it in dimensionless form.
c) What is the change in $\dot{m}$ if the temperature is raised and the water's viscosity drops by a factor of two?

**Solution 9.10**. a) This part is just dimensional analysis. The solution parameter is $\dot{m}$, and boundary condition & material parameters are: $\rho$ = density of the fluid, $\mu$ = viscosity of the fluid, $g$ = acceleration of gravity, $d$ = tube diameter, and $L$ = tube length. The parameter matrix is:

| | $\dot{m}$ | ρ | $\mu$ | g | d | L |
|---|---|---|---|---|---|---|
| Mass: | 1 | 1 | 1 | 0 | 0 | 0 |
| Length: | 0 | -3 | -1 | 1 | 1 | 1 |
| Time: | -1 | 0 | -1 | -2 | 0 | 0 |

and it has rank 3. The number of dimensionless groups is: 6 parameters - 3 dimensions = 3.
These groups are: $\Pi_1 = \dfrac{\dot{m}}{\mu d}$, $\Pi_2 = \dfrac{g\rho^2 d^3}{\mu^2}$, and $\Pi_3 = \dfrac{L}{d}$. Therefore, the scaling law is:

$$\frac{\dot{m}}{\mu d} = F\left(\frac{g\rho^2 d^3}{\mu^2}, \frac{L}{d}\right).$$

b) When the velocity profile in the tube is steady and independent of the vertical coordinate, the vertical momentum equation is

$$0 = -\rho g + \frac{\mu}{R}\frac{\partial}{\partial R}\left(R\frac{\partial u_z}{\partial R}\right)$$

where the $z$-axis points upward (opposite gravity), $R$ is the radial coordinate, and $u_z$ is the vertical velocity. This is the equation for Poiseuille flow with the pressure gradient replaced by $\rho g$. Noting that downward velocity produces positive $\dot{m}$, the results in Section 8.2 imply:

$$\dot{m} = \rho Q = \rho\frac{\pi}{8\mu}\left(\frac{d}{2}\right)^4 \rho g = \frac{\pi}{128}\frac{\rho^2 g d^4}{\mu} \quad \text{or} \quad \frac{\dot{m}}{\mu d} = \frac{\pi}{128}\frac{\rho^2 g d^3}{\mu^2} \rightarrow \Pi_1 = \frac{\pi}{128}\Pi_2.$$

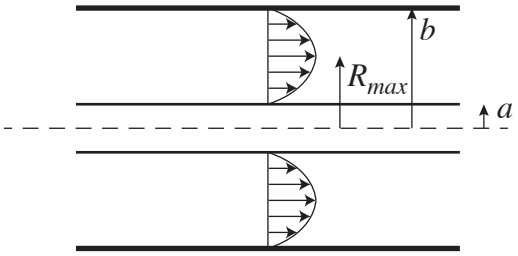
c) The flow rate is inversely proportional to $\mu$ and $\mu = \mu(T)$. Since the ratio $\dfrac{\mu_{room\ temp}}{\mu_{hotter}} = 2$, the flow rate will approximately double (the density of water falls slightly as $T$ increases toward the boiling point) but this will likely be a very small effect.

**Exercise 9.11**. Consider steady laminar flow through the annular space formed by two coaxial tubes aligned with the $z$-axis. The flow is along the axis of the tubes and is maintained by a pressure gradient $dp/dz$. Show that the axial velocity at any radius $R$ is

$$u_z(R) = \frac{1}{4\mu}\frac{dp}{dz}\left[R^2 - a^2 - \frac{b^2 - a^2}{\ln(b/a)}\ln\frac{R}{a}\right],$$

where $a$ is the radius of the inner tube and $b$ is the radius of the outer tube. Find the radius at which the maximum velocity is reached, the volume flow rate, and the stress distribution.

**Solution 9.11**. Use cylindrical coordinates, and assume $u_z = u_z(R)$, where $R$ is the radial coordinate. The axial ($z$-direction) momentum equation is:




$$0 = \frac{\partial p}{\partial z} + \mu\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial u_z}{\partial R}\right).$$

Integrate twice to find:

$$R\frac{\partial u_z}{\partial R} = \frac{R^2}{2\mu}\frac{\partial p}{\partial z} + A \ , \text{ and } \ u_z(R) = \frac{R^2}{4\mu}\frac{\partial p}{\partial z} + A\log R + B.$$

The two boundary conditions are $u = 0$ at $R = a, b$ and these allow the two constants $A$ and $B$ to be determined. Alternatively evaluating the above formula for $u_z(R)$ at $R = a$ and $b$ produces.

$$u_z(a) = 0 = \frac{a^2}{4\mu}\frac{\partial p}{\partial z} + A\log a + B, \text{ and } \ u_z(b) = \frac{b^2}{4\mu}\frac{\partial p}{\partial z} + A\log b + B.$$

These two equations can be solved to find:

$$A = -\frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\frac{b^2 - a^2}{\ln(b/a)}, \text{ and } \ B = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\left[\frac{b^2 - a^2}{\ln(b/a)}\log(a) - a^2\right],$$

so that:

$$u(R) = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\left[R^2 - a^2 - \frac{b^2 - a^2}{\ln(b/a)}\log(R/a)\right].$$

The shear stress is:

$$\tau = \mu\frac{\partial u}{\partial R} = \frac{R}{2\mu}\frac{\partial p}{\partial z} - \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\frac{b^2 - a^2}{R\ln(b/a)}.$$

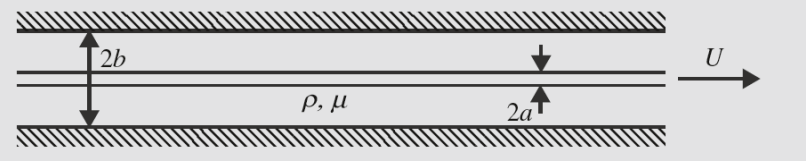
The maximum velocity occurs where $\tau = 0$, and this condition implies:

$$0 = \frac{R_{\max}}{2\mu}\frac{\partial p}{\partial z} - \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\frac{b^2 - a^2}{R_{\max}\ln(b/a)} \ \text{ or } \ R_{\max} = \sqrt{\frac{b^2 - a^2}{2\ln(b/a)}}.$$

The volume flow rate $Q$ is:

$$Q = 2\pi\int_a^b u(R)RdR = \frac{\pi}{8\mu}\left(\frac{\partial p}{\partial z}\right)\left[\frac{\left(b^2 - a^2\right)^2}{\ln(b/a)} + a^4 - b^4\right].$$

**Exercise 9.12**. A long round wire with radius $a$ is pulled at a steady speed $U$, along the axis of a long round tube of radius $b$ that is filled with a viscous fluid. Assuming laminar fully-developed axial flow with $\partial p/\partial z = 0$ in cylindrical coordinates $(R,\varphi,z)$ with $\mathbf{u} = (0, 0, w(R))$, determine $w(R)$ assuming constant fluid density $\rho$ and viscosity $\mu$ with no body force.
a) What force per unit length of the wire is needed to maintain the motion of the wire?
b) Explain what happens to $w(R)$ when $b \to \infty$. Is this situation physically meaningful? What additional term(s) from the equations of motion need to be retained to correct this situation?




**Solution 9.12**. The steady axisymmetric no-swirl equations are:

$$\frac{1}{R}\frac{\partial}{\partial R}\left(Ru_R\right) + \frac{\partial w}{\partial z} = 0,\ u_R\frac{\partial u_R}{\partial R} + w\frac{\partial u_R}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial R} + \frac{\mu}{\rho}\left(\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial u_R}{\partial R}\right) + \frac{\partial^2 u_R}{\partial z^2} - \frac{u_R}{R^2}\right),$$

$$0 = -\frac{1}{\rho R}\frac{\partial p}{\partial \varphi},\ \text{and}\ u_R\frac{\partial w}{\partial R} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\left(\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial w}{\partial R}\right) + \frac{\partial^2 w}{\partial z^2}\right).$$

Setting $\mathbf{u} = (0, 0, w(R))$ automatically satisfies the cons. of mass equation. With $\partial p/\partial z = 0$, the other equations further simplify to:

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial R} = -\frac{1}{\rho R}\frac{\partial p}{\partial \varphi},\ \text{and}\ 0 = \frac{\mu}{\rho}\left(\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial w}{\partial R}\right)\right)$$

Thus, the pressure is constant everywhere because its gradient is zero. For nonzero viscosity and density, integrate the final equation once to find: $C_1 = R\left(\partial w/\partial R\right)$ where $C_1$ is a constant. Divide by $R$ and integrate again: $C_1\ln(R) + C_2 = w(R)$ where $C_2$ is a constant. The two constants can be determined from the boundary conditions on the wire and tube surfaces:

$$w(R = a) = U = C_1\ln(a) + C_2,\ \text{and}\ w(R = b) = 0 = C_1\ln(b) + C_2.$$

Simultaneous solution of these two algebraic equations produces: $w(R) = \dfrac{U}{\ln\left(b/a\right)}\ln\left(b/R\right)$

a) The viscous shear stress on the wire, $\tau_w$, will act to retard its motion. The viscous force per unit length of wire will act in the $z$-direction on the wire and will be

$$2\pi a\tau_w = 2\pi a\mu\left(\frac{\partial w}{\partial R}\right)_{R=a} = 2\pi a\mu\left(\frac{U}{\ln\left(b/a\right)}\frac{1}{b/R}\left(-\frac{b}{R^2}\right)\right)_{R=a} = -\frac{2\pi\mu U}{\ln\left(b/a\right)}.$$

Thus, the force necessary to maintain the motion will be: $= +2\pi\mu U\mathbf{e}_z/\ln\left(b/a\right)$
b) When $b \to \infty$, the solution indicates that the force necessary to keep the wire moving will approach zero. This is not physically meaningful for any finite wire length and time duration because the fluid is viscous. The problem(s) with this limiting situation can be corrected by including the unsteady terms, $\partial/\partial t$, or terms involving $u_R$ and $\partial/\partial z$ in the conservation equations.

**Exercise 9.13**. Consider steady unidirectional incompressible viscous flow in Cartesian coordinates, $u = v = 0$ with $w = w(x,y)$ without body forces.
a) Starting from the steady version of (8.1), derive a single equation for $w$ assuming that $\partial p/\partial z$ is non-zero and constant.
b) Guess $w(x,y)$ for a tube with elliptical cross section $(x/a)^2 + (y/b)^2 = 1$.
c) Determine $w(x,y)$ in for a tube of rectangular cross section: $-a/2 \le x \le +a/2$, $-b/2 \le y \le +b/2$. (Hint: find particular (a polynomial) and homogeneous (a Fourier series) solutions for $w$.)

**Solution 9.13**. a) Start with the full $z$-direction momentum equation in Cartesian coordinates:

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right).$$

For steady flow with $u = v = 0$ and $w = w(x,y)$, a lot terms that drop out of the above equation, and it simplifies to: $\dfrac{\partial p}{\partial z} = \mu\left(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}\right)$. Note that the density has dropped out because there is no fluid acceleration in this type of flow.
b) The boundary condition for this flow is $w = 0$ on the wall of the tube. For an elliptical cross section given by $(x/a)^2 + (y/b)^2 = 1$, a possible solution is $w(x,y) = W_o\left[1 - (x/a)^2 - (y/b)^2\right]$ where $W_o$ is a constant that needs to be determined from the equation of motion. Plugging this guessed solution into the equation found for part a), produces: $\dfrac{\partial p}{\partial z} = -\mu W_o\left(\dfrac{2}{a^2} + \dfrac{2}{b^2}\right)$. If $\partial p/\partial z$ is constant then: $w(x,y) = -\dfrac{1}{2\mu}\dfrac{\partial p}{\partial z}\left(\dfrac{1}{a^2} + \dfrac{1}{b^2}\right)^{-1}\left[1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}\right]$, and this reduces to the usual Poiseuille flow formula when $a = b$.
c) The solution for a rectangular cross section is a bit more involved. The equation for $w$ in part a) is inhomogeneous, so its solution will be made up of a particular solution and a homogeneous solution: $w = w_p + w_h$. Here, the particular solution must satisfy the governing equation, but it need not satisfy the boundary conditions. The constants in the homogeneous solution can be used to fix up the solution at the tube walls. Therefore, start by looking for a one-variable particular solution, i.e. $w_p = w_p(x)$, and plug this into the field equation:

$$\frac{\partial p}{\partial z} = \mu\left(\frac{\partial^2 w_p}{\partial x^2} + \frac{\partial^2 w_p}{\partial y^2}\right) = \mu\frac{d^2 w_p}{dx^2}.$$

Integrate twice to find: $w_p(x) = \dfrac{1}{2\mu}\dfrac{dp}{dz}x^2 + C_1 x + C_2$. The two constants can be chosen to zero-out the velocity at $x = \pm a/2$; $w_p(x) = -\dfrac{a^2}{8\mu}\dfrac{dp}{dz}\left(1 - \dfrac{x^2}{(a/2)^2}\right)$. Now, the homogeneous solution is needed. It solves $\dfrac{\partial^2 w_h}{\partial x^2} + \dfrac{\partial^2 w_h}{\partial y^2} = 0$. Assume a variables-separable form: $w_h(x,y) = X(x)Y(y)$, plug it in, and label the separation constant $-k^2$ to find two ordinary differential equations: $X'' + k^2 X = 0$, and $Y'' - k^2 Y = 0$. The symmetry of the situation implies that only the cos-part of $X$ and the cosh-part of $Y$ are needed: $w_h(x,y) = A\cos(kx)\cosh(ky)$. To zero-out the velocity at $x$

$= \pm a/2$, the separation constant $k$ must be restricted to a set of special values: $k = k_n = (2n-1)\pi/a$. Thus, the overall solution takes the following form:

$$w = w_p + w_h = -\frac{a^2}{8\mu}\frac{dp}{dz}\left(1-\frac{x^2}{(a/2)^2}\right) + \sum_{n=1}^{\infty} A_n \cos\left(\frac{(2n-1)\pi x}{a}\right)\cosh\left(\frac{(2n-1)\pi y}{a}\right). \qquad (1)$$

Here it is clear than $w$ is zero on $x = \pm a/2$. The $A_n$'s are determined by requiring $w = 0$ on $y = \pm b/2$:

$$\frac{a^2}{8\mu}\frac{dp}{dz}\left(1-\frac{x^2}{(a/2)^2}\right) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{(2n-1)\pi x}{a}\right)\cosh\left(\frac{(2n-1)\pi b}{2a}\right).$$

Multiply both sides by $\cos[(2m-1)\pi x/a]$, and integrate from $x = -a/2$ to $x = +a/2$:

$$\frac{a^2}{8\mu}\frac{dp}{dz}\int_{x=-a/2}^{x=a/2}\left(1-\frac{x^2}{(a/2)^2}\right)\cos\left(\frac{(2m-1)\pi x}{a}\right)dx = A_m \cosh\left(\frac{(2m-1)\pi b}{2a}\right)\frac{a}{2}$$

The integral on the left side is a little tedious but can be evaluated by parts:

$$\frac{a^2}{8\mu}\frac{dp}{dz}\left((-1)^{m+1}\frac{16a}{(2m-1)^3\pi^3}\right) = A_m \cosh\left(\frac{(2m-1)\pi b}{2a}\right)\frac{a}{2}$$

Thus: $A_m = \dfrac{4a^2}{\mu}\dfrac{dp}{dz}\left(\dfrac{(-1)^{m+1}}{(2m-1)^3\pi^3}\right)\Big/\cosh\left(\dfrac{(2m-1)\pi b}{2a}\right)$. This result together with (1) completes the solution.

**Exercise 9.14**. A long vertical cylinder of radius $b$ rotates with angular velocity $\Omega$ concentrically outside a smaller stationary cylinder of radius $a$. The annular space is filled with fluid of viscosity $\mu$. Show that the steady velocity distribution is: $u_\varphi = \dfrac{R^2 - a^2}{b^2 - a^2}\dfrac{b^2\Omega}{R}$, and that the torque exerted on either cylinder, per unit length, equals $4\pi\mu\Omega a^2b^2/(b^2 - a^2)$.

**Solution 9.14**. Use cylindrical coordinates, and start from equation (8.10),

$$u_\varphi(R) = \frac{1}{R_2^2 - R_1^2}\left\{\left[\Omega_2 R_2^2 - \Omega_1 R_1^2\right]R - \left[\Omega_2 - \Omega_1\right]\frac{R_1^2 R_2^2}{R}\right\},$$

and set $R_1 = a$, $R_2 = b$, $\Omega_2 = \Omega$, and $\Omega_1 = 0$ to find:

$$u_\varphi(R) = \frac{1}{b^2 - a^2}\left\{\left[\Omega b^2 - 0\right]R - \left[\Omega - 0\right]\frac{a^2b^2}{R}\right\} = \frac{\Omega b^2}{b^2 - a^2}\frac{R^2 - a^2}{R} = \frac{R^2 - a^2}{b^2 - a^2}\frac{b^2\Omega}{R}.$$

From Appendix B the shear stress is

$$\tau_{R\varphi} = \mu R\frac{\partial}{\partial R}\left(\frac{u_\varphi}{R}\right) = \frac{\mu R b^2\Omega}{b^2 - a^2}\frac{\partial}{\partial R}\left(1 - \frac{a^2}{R^2}\right) = \frac{2\mu a^2b^2\Omega}{b^2 - a^2}\left(\frac{1}{R^2}\right).$$

If the axial length of the cylinders is $h$, then

torque on the inner cylinder = $(\tau_{R\varphi})_{R=a}$(area)(radius) = $\dfrac{2\mu a^2b^2\Omega}{b^2 - a^2}\left(\dfrac{1}{a^2}\right)(2\pi ah)a = \dfrac{4\pi\mu a^2b^2h\Omega}{b^2 - a^2}$,

torque on the outer cylinder = $(\tau_{R\varphi})_{R=b}$(area)(radius) = $\dfrac{2\mu a^2b^2\Omega}{b^2 - a^2}\left(\dfrac{1}{b^2}\right)(2\pi bh)b = \dfrac{4\pi\mu a^2b^2h\Omega}{b^2 - a^2}$.

**Exercise 9.15**. Consider a solid cylinder of radius $a$, steadily rotating at angular speed $\Omega$ in an infinite viscous fluid. The steady solution is irrotational: $u_\theta = \Omega a^2/R$. Show that the work done by the external agent in maintaining the flow (namely, the value of $2\pi R u_\theta \tau_{r\theta}$ at $R = a$) equals the viscous dissipation rate of fluid kinetic energy in the flow field.

**Solution 9.15**. Using the given velocity field, the shear stress is:

$$\tau_{R\varphi} = \mu R \frac{\partial}{\partial R}\left(\frac{u_\varphi}{R}\right) = \mu \Omega a^2 R \frac{\partial}{\partial R}\left(\frac{1}{R^2}\right) = -2\mu\Omega a^2 \frac{1}{R^2}.$$

The work done per unit height = $\left\{2\pi a \tau_{R\varphi} u_\varphi\right\}_{R=a} = 2\pi a \cdot 2\mu\Omega \cdot \Omega a = 4\pi\mu a^2\Omega^2$.

From (4.58) the viscous dissipation rate of kinetic energy per unit volume for an incompressible flow is $\rho\varepsilon = 2\mu S_{ij} S_{ij}$, where $\varepsilon$ is the viscous dissipation of kinetic energy per unit mass. For the given flow field there is only one non-zero independent strain component:

$$S_{R\varphi} = S_{\varphi R} = \frac{R}{2}\frac{\partial}{\partial R}\left(\frac{u_\varphi}{R}\right) = \frac{\Omega a^2}{2} R \frac{\partial}{\partial R}\left(\frac{1}{R^2}\right) = -\Omega a^2 \frac{1}{R^2}.$$

Therefore:

$$\rho\varepsilon = 2\mu S_{ij} S_{ij} = 2\mu\left(S_{R\varphi}^2 + S_{\varphi R}^2\right) = 4\mu\Omega^2 \frac{a^4}{R^4},$$

so the kinetic energy dissipation rate per unit height is:

$$\int_a^\infty \rho\varepsilon 2\pi R dR = 8\pi\mu\Omega^2 a^4 \int_a^\infty \frac{1}{R^3} dR = 4\pi\mu\Omega^2 a^2,$$

which equals the work done turning the cylinder.

**Exercise 9.16**. Redo the lubrication-theory scaling provided in Section 9.3 for the situation when there is an imposed external time scale $\tau$ so that the appropriate dimensionless time is $t^* = t/\tau$, instead of that shown in (9.14). Show that this leads to the additional requirement $\rho h^2/\mu\tau << 1$ for the validity of (9.17a). Interpret this new requirement in terms of the viscous diffusion length $\sqrt{\nu\tau}$ . Is the unsteady acceleration term needed to analyze the effect of a 100 Hz oscillation imposed on a 0.1-mm film of 30-weight oil ($\nu \approx 4\ \mathrm{x}10^{-4}$ m$^2$/s)?

**Solution 9.16**. Use the dimensionless scaling provided by (9.14) except replace that used for the time variable with $t^* = t/\tau$. The remaining members of (9.14),

$$x^* = x/L\ ,\ y^* = y/h = y/\varepsilon L,\ u^* = u/U\ ,\ v^* = v/\varepsilon U\ ,\ \text{and}\ p^* = p/P_a,\ \text{where}\ \varepsilon = h/L,$$

are unchanged. This mildly revised scaling does not influence the continuity equation. However, it does impact the unsteady term in both momentum equations, but it is only necessary to consider one. The horizontal momentum equation is the more interesting equation. Substituting in the dimensionless variables leads to:

$$\frac{\rho U}{\tau}\frac{\partial u^*}{\partial t^*} + \left(\frac{\rho U^2}{L}\right)u^*\frac{\partial u^*}{\partial x^*} + \left(\frac{\varepsilon\rho U^2}{\varepsilon L}\right)v^*\frac{\partial u^*}{\partial y^*} = -\left(\frac{P_a}{L}\right)\frac{\partial p^*}{\partial x^*} + \mu\left(\frac{U}{L^2}\right)\frac{\partial^2 u^*}{\partial x^{*2}} + \mu\left(\frac{U}{\varepsilon^2 L^2}\right)\frac{\partial^2 u^*}{\partial y^{*2}}\ .$$

Dividing by $\mu U/\varepsilon^2 L^2$, produces:

$$\frac{\rho\varepsilon^2 L^2}{\mu\tau}\frac{\partial u^*}{\partial t^*} + \left(\varepsilon^2 R_e\right)u^*\frac{\partial u^*}{\partial x^*} + \left(\varepsilon^2 R_e\right)v^*\frac{\partial u^*}{\partial y^*} = -\left(\frac{1}{\Lambda}\right)\frac{\partial p^*}{\partial x^*} + \varepsilon^2\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\ .$$

The coefficient of the first term is different than that given in (9.16a), and it must be small for the validity of (9.17a):

$$\frac{\rho\varepsilon^2 L^2}{\mu\tau} = \frac{\rho h^2}{\mu\tau} = \frac{h^2}{\nu\tau} << 1 \quad \text{or} \quad h << [\nu\tau]^{1/2}.$$

where $\nu = \mu/\rho$. This new requirement states that the diffusion distance, $[\nu\tau]^{1/2}$, associated with the time scale of flow unsteadiness $\tau$ must much larger than the gap distance $h$.

For a 100 Hz oscillation ($t \sim 10$ ms) imposed on a 0.1 mm gap containing 30-weight oil with $\nu \approx 4\ \mathrm{x}10^{-4}$ m$^2$/s:

$$\frac{h^2}{\nu\tau} = \frac{(10^{-4}m)^2}{(4\times10^{-4}m^2s^{-1})(0.01s)} = 0.0025\,.$$

This is much less than unity, so the unsteady term is not needed in an elementary lubrication analysis of this situation. However, if the gap were 10 times larger (1 mm) this conclusion would be different.

**Exercise 9.17**. For lubrication flow under the sloped bearing of Example 9.3, the assumed velocity profile was $u(x,y) = -(1/2\mu)(dP/dx)y(h(x)-y)) + Uy/h(x)$, the derived pressure was $P(x) = P_e + \left(3\mu U\alpha/h_o^2 L\right)x(L-x)$, and the load (per unit depth) carried by the bearing was $W = \mu U\alpha L^2/2h_o^2$. Use these equations to determine the frictional force (per unit depth), *Ff*, applied to the lower (flat) stationary surface in terms of *W*, $h_o/L$, and $\alpha$. What is the spatially-averaged coefficient of friction under the bearing?

**Solution 9.17**. For lubrication flow under the sloped bearing, the velocity field was assumed to be: $u(y) = -\dfrac{1}{2\mu}\dfrac{\partial p}{\partial x}y(h-y) + \dfrac{Uy}{h}$, so the wall shear stress is: $\tau_w = \left(\mu\dfrac{\partial u}{\partial y}\right)_{y=0} = -\dfrac{h}{2}\dfrac{\partial p}{\partial x} + \mu\dfrac{U}{h}$.

Integrate this from $x = 0$ to $x = L$ to find the friction force $F_f$ per unit depth. Here the pressure gradient will be: $dp/dx = \left(3\mu U\alpha/h_o^2 L\right)(L-2x)$, so

$$F_f = \int_0^L \tau_w dx = \int_0^L \left(-\frac{h_o(1-\alpha x/L)}{2}\frac{3\mu U\alpha}{h_o^2 L}(L-2x) + \mu\frac{U}{h_o(1-\alpha x/L)}\right)dx$$

Both integrals are elementary, so $F_f$ is found to be: $F_f = -\dfrac{\mu U\alpha^2 L}{4h_o} - \dfrac{\mu UL}{h_o\alpha}\ln(1-\alpha)$.

However, because $\alpha$ is small, only the $\alpha^0$ and $\alpha^1$ terms should be reported so, $F_f \cong \dfrac{\mu UL}{h_o}\left(1+\dfrac{\alpha}{2}\right)$,

or using $W = \dfrac{\mu U\alpha L^2}{2h_o^2}$, $F_f \cong \dfrac{Wh_o}{\alpha L}(2+\alpha)$. The coefficient of friction is a ratio:

$$\frac{F_f}{W} \cong \frac{h_o(2+\alpha)}{\alpha L} \approx \frac{2h_o}{\alpha L}.$$

Hence, we can see that it is not possible to obtain a small coefficient of friction since the maximum value of $\alpha$ is ~$h_o/L$. Thus, the purpose of this type of bearing would not necessarily be to produces a small coefficient of friction. Instead it prevents scuffing and wear since the presence of the oil film prevents the bearing pad and the surface from coming into contact.

**Exercise 9.18**. A bearing pad of total length $2L$ moves to the right at constant speed $U$ above a thin film of incompressible oil with viscosity $\mu$ and density $\rho$. There is a step change in the gap thickness (from $h_1$ to $h_2$) below the bearing as shown. Assume that the oil flow under the bearing pad follows: $u(y) = -\dfrac{y(h_j - y)}{2\mu}\dfrac{dP(x)}{dx} + \dfrac{Uy}{h_j}$, where $j$ = 1 or 2. The pad is instantaneously aligned above the coordinate system shown. The pressure in the oil ahead and behind the bearing is $P_e$.

a) By conserving mass for the oil flow, find a relationship between $\mu$, $U$, $h_j$, $dP/dx$, and an unknown constant $C$.

b) Use the result of part a) and continuity of the pressure at $x = 0$, to determine

$$P(0) - P_e = \frac{6\mu UL(h_1 - h_2)}{h_2^3 + h_1^3}$$

c) Can this bearing support a externally applied downward load when $h_1 < h_2$?

**Solution 9.18**. a) Use a CV attached to the bearing pad with vertical flux surfaces that span the gap. Conservation of mass implies:

$$\int_0^{h_j} \big(u(y) - U\big)dy = C \;\rightarrow\; \int_0^{h_j}\left(-\frac{y(h_j - y)}{2\mu}\frac{dP(x)}{dx} + \frac{Uy}{h_j} - U\right)dy = C$$

Performing the integration produces:

$$-\frac{1}{2\mu}\frac{dP(x)}{dx}\left(\frac{h_j^3}{2} - \frac{h_j^3}{3}\right) + U\frac{h_j}{2} - Uh_j = -\frac{h_j^3}{12\mu}\frac{dP(x)}{dx} - \frac{Uh_j}{2} = C\,.$$

b) Use part a), solve for $dP/dx$,

$$\frac{dP(x)}{dx} = -\frac{12\mu}{h_j^3}\left(C + \frac{Uh_j}{2}\right), \text{ and integrate to find: } P(x) - P(0) = -\frac{12\mu x}{h_j^3}\left(C + \frac{Uh_j}{2}\right).$$

Evaluate $P(x)$ at $x = \pm L$ and note that $P(+L) = P(-L) = P_e$ with $P(x)$ continuous at $x = 0$:

$$-\frac{12\mu L}{h_1^3}\left(C + \frac{Uh_1}{2}\right) = +\frac{12\mu L}{h_2^3}\left(C + \frac{Uh_2}{2}\right).$$

Solve for $C = -\dfrac{Uh_1h_2}{2}\left\{\dfrac{h_1^2 + h_2^2}{h_1^3 + h_2^3}\right\}$, and then find:

$$P(0) - P_e = \frac{12\mu L}{h_1^3}\left(C + \frac{Uh_1}{2}\right) = \frac{12\mu L}{h_1^3}\left(-\frac{Uh_1h_2}{2}\left\{\frac{h_1^2 + h_2^2}{h_1^3 + h_2^3}\right\} + \frac{Uh_1}{2}\right) = \frac{6\mu UL}{h_1^2}\left(-h_2\left\{\frac{h_1^2 + h_2^2}{h_1^3 + h_2^3}\right\} + 1\right)$$

$$= \frac{6\mu UL}{h_1^2}\left(\frac{-h_1^2h_2 - h_2^3 + h_1^3 + h_2^3}{h_1^3 + h_2^3}\right) = \frac{6\mu UL}{h_1^2}\left(\frac{-h_1^2h_2 + h_1^3}{h_1^3 + h_2^3}\right) = 6\mu UL\left(\frac{h_1 - h_2}{h_1^3 + h_2^3}\right)$$

c) When $h_1 < h_2$, the pressure at $x = 0$ is less than $P_e$. This means that the bearing pad is sucked toward the surface, so when $h_1 < h_2$ the bearing cannot support an external load.

**Exercise 9.19**. A flat disk of radius $a$ rotates above a solid boundary at a steady rotational speed of $\Omega$. The gap, $h$ ($<< a$), between the disk and the boundary is filled with an incompressible Newtonian fluid with viscosity $\mu$ and density $\rho$. The pressure at the edge of the disk is $p(a)$.
a) Using cylindrical coordinates and assuming that the only non-zero velocity component is $u_\varphi(R,z)$, determine the torque necessary to keep the disk turning.
b) If $p(a)$ acts on the exposed (upper) surface of the disk, will the pressure distribution on the disk's wetted surface tend to pull the disk toward or push it away from the solid boundary?
c) If the gap is increased, eventually the assumption of part a) breaks down. What happens? Explain why and where $u_r$ and $u_z$ might be non-zero when the gap is no longer narrow.

**Solution 9.19**. a) By analogy with plane Couette flow, set $u_\varphi(R,z) = \Omega R(z/h)$. The shear stress on any element of the rotating disk will be: $\tau = \mu(du_\varphi/dz) = \mu\Omega R/h$. The torque necessary to keep the disk turning will be:

$$Torque = \int_{r=0}^{r=a} R\tau dA = 2\pi(\mu\Omega/h)\int_{r=o}^{r=a} R^3 dR = \pi\mu\Omega a^4/2h\,.$$

b) For the assumed solution used to determine the answer to part a) to be valid, the quadratic velocity terms in the conservation of momentum equation must approach zero. Thus, the radial momentum equation: $-\rho\dfrac{u_\varphi^2}{R} \approx -\dfrac{\partial p}{\partial R}$ suggests that the radial pressure gradient will be very small but positive because $u_\varphi$ is non-zero just underneath the rotating disk. Thus, the pressure near the axis of rotation may be slightly lower. This suggests the disk may be mildly drawn downward as it rotates.
c) If the gap increases in size, a secondary flow with closed streamlines will develop that has fluid moving away from the axis of rotation near the spinning disk, downward as the fluid approaches $R = a$, toward the axis of rotation near the stationary disk, and upward near the axis of rotation. This secondary flow will be driven by the radial pressure gradient mentioned above in part b).

**Exercise 9.20**. A circular block with radius $a$ and weight $W$ is released at $t = 0$ on a thin layer of an incompressible fluid with viscosity $\mu$ that is supported by a smooth horizontal motionless surface. The fluid layer's initial thickness is $h_o$. Assume that flow in the gap between the block and the surface is quasi-steady with a parabolic velocity profile:

$$u_R(R,z,t) = -\big(dP(R)/dR\big)z\big(h(t)-z\big)\big/2\mu$$

where $R$ is the distance from the center of the block, $P(R)$ is the pressure at $R$, $z$ is the vertical coordinate from the smooth surface, $h(t)$ is the gap thickness, and $t$ is time.

a) By considering conservation of mass, show that: $dh/dt = (h^3/6\mu R)(dP/dR)$.

b) If $W$ is known, determine $h(t)$ and note how long it takes for $h(t)$ to reach zero.

**Solution 9.20**. Consider a circular control volume under the block with radius of $R$ and height $h(t)$. The integral form of conservation of mass implies:

$$\pi R^2\rho\frac{dh}{dt} = -2\pi R\rho\int_0^{h(t)} u_R(R,z,t)dz = 2\pi R\rho\int_0^{h(t)}\frac{1}{2\mu}\frac{dP(R)}{dR}z\big(h(t)-z\big)\big)dz = \frac{\pi R\rho}{\mu}\frac{dP(R)}{dR}\frac{h^3}{6}$$

which simplifies to: $\dfrac{dh}{dt} = \dfrac{h^3}{6\mu R}\dfrac{dP(R)}{dR}$.

b) Since $h$ does not depend on $R$, this equation can be integrated in the radial direction from $R$ to $a$:

$$\frac{dh}{dt}\int_R^a RdR = \frac{h^3}{6\mu}\int_R^a dP \quad \text{which implies} \quad \frac{dh}{dt}\left(\frac{a^2-R^2}{2}\right) = \frac{h^3}{6\mu}\big(P_e - P(R)\big),$$

where $P_e$ is the exterior pressure on the circular block (perhaps equal to 1 atm). Assuming quasi-steady conditions, the weight of the block must be balanced by an integral of the excess-pressure underneath the block:

$$W = 2\pi\int_0^a \big(P(R)-P_e\big)RdR = -\frac{dh}{dt}\frac{6\pi\mu}{h^3}\int_0^a\big(a^2-R^2\big)RdR = -\frac{dh}{dt}\frac{6\pi\mu}{h^3}\left(a^2\frac{a^2}{2}-\frac{a^4}{4}\right) = -\frac{dh}{dt}\frac{3\pi\mu a^4}{2h^3}.$$

Now, treat this result as a first-order nonlinear differential equation for $h(t)$ and integrate in time:

$$\int_{h_o}^{h(t)} h^{-3}dh = -\frac{1}{2}\left(\frac{1}{h^2}-\frac{1}{h_o^2}\right) = -\frac{2Wt}{3\pi\mu a^4} \quad \text{which implies} \quad \frac{1}{h^2(t)} = \frac{1}{h_o^2}+\frac{4Wt}{3\pi\mu a^4}.$$

Solving for $h(t)$ yields: $h(t) = h_o\left[1+\dfrac{4Wh_o^2t}{3\pi\mu a^4}\right]^{-1/2}$. For this solution, $h(t) \to 0$ only when $t \to \infty$.

**Exercise 9.21**. Consider the inverse of the previous exercise. A block and smooth surface are separated by a thin layer of a viscous fluid with thickness $h_o$. At $t = 0$, a force, $F$, is applied to separate them. If $h_o$ is arbitrarily small can the block and plate be separated easily? Perform some tests in your kitchen. Use maple syrup, peanut butter, liquid soap, pudding, etc. for the viscous liquid. The flat top-side of a metal jar lid or the flat bottom of a drinking glass make a good circular block. (Lids with raised edges and cups & glasses with ridges or sloped bottoms do not work well). A flat countertop or the flat portion of a dinner plate can be the motionless smooth surface. Can the item used for the block be more easily separated from the surface when tilted relative to the surface? Describe your experiments and try to explain your results.

**Solution 9.21**. Using a Formica counter-top and the underside of a coffee-cup saucer, a compelling case can be made for the accuracy of the result of Exercise 9.20. If an upward force $F$ is applied to the flat-bottomed block, the gap distance as a function of time is:

$$h(t) = h_o\left[1 - \frac{4Fh_o^2 t}{3\pi\mu a^4}\right]^{-1/2}$$

(this is the same as the answer for Exercise 9.20 with $W$ replaced by $-F$). This formula then predicts that the time for the plate to rise to $z = +\infty$, $t_\infty$, is:

$$t_\infty = \frac{3\pi\mu a^4}{4Fh_o^2}.$$

So, $t_\infty$ will increase with increasing $a$ and $\mu$, and decrease with increasing $F$ and $h_o$. Puddles of peanut butter and pancake syrup both resisted vertical motion of the saucer. The peanut butter (more viscous) was more effective in preventing the saucer from being lifted. The other parametric dependencies were qualitatively verified as well. In all cases, the adhesion between the saucer and the countertop failed when a small air-passage opened up under the saucer. This allowed air, which is much less viscous than the liquids tested, to rush in under the saucer allowing it to be lifted.

**Exercise 9.22**. A rectangular slab of width $2L$ (and depth $B$ into the page) moves *vertically* on a thin layer of oil that flows horizontally as shown. Assume $u(y,t) = -\left(h^2/2\mu\right)\left(dP/dx\right)\left(y/h\right)\left(1-y/h\right)$ where $h(x,t)$ is the instantaneous gap between the slab and the surface, $\mu$ is the oil's viscosity, and $P(x,t)$ is the pressure in the oil below the slab. The slab is slightly misaligned with the surface so that $h(x,t) = h_o\left(1+\alpha\, x/L\right) + \dot{h}_o t$ where $\alpha << 1$ and $\dot{h}_o$ is the vertical velocity of the slab. The pressure in the oil outside the slab is $P_o$. Consider the instant $t = 0$ in your work below.

a) Conserve mass in an appropriate CV to show that: $\dfrac{\partial h}{\partial t} + \dfrac{\partial}{\partial x}\left(\int_0^{h(x,t)} u(y,t)dy\right) = 0$.

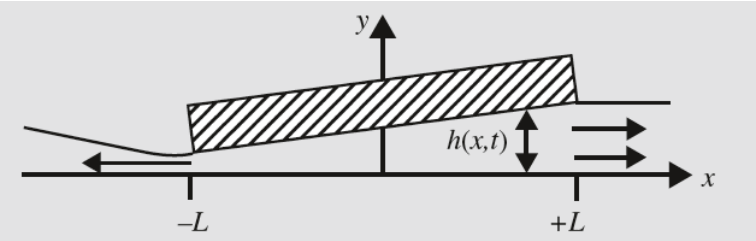
b) Keeping only linear terms in $\alpha$, and noting that $C$ and $D$ are constants, show that:

$$P(x,t=0) = \frac{12\mu}{h_o^3}\left(\frac{\dot{h}_o x^2}{2}\left(1-\frac{2\alpha x}{L}\right) + Cx\left(1-\frac{3\alpha x}{2L}\right)\right) + D.$$

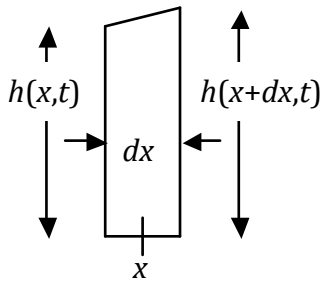
c) State the boundary conditions necessary to evaluate the constants $C$ and $D$.

d) Evaluate the constants to show that the pressure distribution below the slab is: $P(x,t) - P_o = -\dfrac{6\mu\dot{h}_o L^2}{h_o^3(t)}\left(1-\left(x/L\right)^2\right)\left(1-2\alpha\dfrac{x}{L}\right)$.

e) Does this pressure distribution act to increase or decrease alignment between the slab and surface when the slab is moving downward? Answer the same question for upward slab motion.




**Solution 9.22**. a) Use the differential CV shown to the right and note that the upper boundary is moving at speed $\mathbf{b} = \left(dh/dt\right)\mathbf{e}_y = \dot{h}_o\mathbf{e}_y$ but there is no mass flux through it. The other boundaries are stationary. Thus, Cons. of Mass,




$\dfrac{\partial}{\partial t}\left(\int_{CV}\rho dV\right) + \int_A \rho\left(\mathbf{u}-\mathbf{b}\right)\cdot\mathbf{n}dA = 0$, implies

$$\frac{\partial}{\partial t}\left(\rho h B dx\right) - \int_0^{h(x,t)} \rho u(x,y,t)Bdy + \int_0^{h(x+dx,t)} \rho u(x+dx,y,t)Bdy = 0$$

Divide each term by $\rho B dx$ to find: $\dfrac{\partial h}{\partial t} - \dfrac{1}{dx}\left[\int_0^{h(x+dx,t)} u(x+dx,y,t)dy - \int_0^{h(x,t)} u(x,y,t)dy\right] = 0$

The stuff inside the [,]/$dx$ term is the definition of $\partial/\partial x$, so: $\dfrac{\partial h}{\partial t} + \dfrac{\partial}{\partial x}\left(\int_0^{h(x,t)} u(y,t)dy\right) = 0$.

b) First compute the integral: $\int_0^h u dy = -\dfrac{h^3}{2\mu}\dfrac{dP}{dx}\int_0^h \dfrac{y}{h}\left(1-\dfrac{y}{h}\right)d\left(\dfrac{y}{h}\right) = -\dfrac{h^3}{12\mu}\dfrac{dP}{dx}$. Put this into the result of part a) to find a differential equation for $P(x,t)$: $\dot{h}_o - \dfrac{\partial}{\partial x}\left(\dfrac{h^3}{12\mu}\dfrac{dP}{dx}\right) = 0$.

Integrate once: $\frac{h^3}{12\mu}\frac{\partial P}{\partial x} = \dot{h}_o x + C$, and then again with $h(x,t)$ evaluated at $t = 0$,

$$P = 12\mu\int \frac{\dot{h}_o x + C}{h_o^3(1+\alpha\, x/L)^3}dx + D \approx \frac{12\mu}{h_o^3}\int \left(\dot{h}_o x + C\right)(1-3\alpha\, x/L)dx + D$$

$$P \approx \frac{12\mu}{h_o^3}\left(\dot{h}_o\frac{x^2}{2} + Cx - \dot{h}_o\frac{3\alpha}{L}\frac{x^3}{3} - C\frac{3\alpha}{L}\frac{x^2}{2}\right) + D = \frac{12\mu}{h_o^3}\left(\dot{h}_o\frac{x^2}{2}\left(1-\frac{2\alpha}{L}x\right) + Cx\left(1-\frac{3\alpha}{2L}x\right)\right) + D$$

Here, $C$ and $D$ could depend on $t$, but the instant of interest is $t = 0$ so this dependence is suppressed.

c) There are two boundary conditions on the pressure at either end of the slab:

$$P(x = -L) = P(x = +L) = P_o.$$

d) Using the results of part b) and c), leads to two equations for the constants $C$ and $D$.

$$P(-L) = P_o = \frac{12\mu}{h_o^3}\left(\dot{h}_o\frac{L^2}{2}(1+2\alpha) - CL\left(1+\frac{3\alpha}{2}\right)\right) + D, \text{ and}$$

$$P(+L) = P_o = \frac{12\mu}{h_o^3}\left(\dot{h}_o\frac{L^2}{2}(1-2\alpha) + CL\left(1-\frac{3\alpha}{2}\right)\right) + D.$$

These can be solved to find:

$$C = \dot{h}_o L\alpha \quad \text{and} \quad D = P_o - \frac{6\mu L^2}{h_o^3}\dot{h}_o,$$

where quadratic terms in $\alpha$ have been dropped. Thus, the pressure distribution is:

$$\begin{aligned} P(x) &= \frac{12\mu}{h_o^3}\left(\dot{h}_o\frac{x^2}{2}\left(1-\frac{2\alpha}{L}x\right) + \dot{h}_o L\alpha x\left(1-\frac{3\alpha}{2L}x\right)\right) + P_o - \frac{6\mu L^2}{h_o^3}\dot{h}_o \\ &= P_o + \frac{6\mu L^2}{h_o^3}\dot{h}_o\left(\frac{x^2}{L^2}\left(1-\frac{2\alpha}{L}x\right) + 2\alpha\frac{x}{L}\left(1-\frac{3\alpha}{2L}x\right) - 1\right) \\ &= P_o - \frac{6\mu L^2}{h_o^3}\dot{h}_o\left(1-\frac{x^2}{L^2} + \frac{2\alpha}{L^3}x^3 - \frac{2\alpha}{L}x + \ldots\right) \\ &= P_o - \frac{6\mu L^2}{h_o^3}\dot{h}_o\left(1-\frac{x^2}{L^2}\right)\left(1-\frac{2\alpha}{L}x\right) + \ldots \end{aligned}$$

where again quadratic terms in $\alpha$ have been dropped.

e) When the slab is moving *downward* (so $\dot{h}_o$ is negative), the highest pressure point lies in the region where the gap is the smallest and this pressure distribution causes a moment on the slab that *tends to align* the slab with the surface. This is beneficial in lubrication flows because the oil pressure acts to prevent contact between slab and the surface. When the slab is moving *upward*, the lowest pressure lies in the region where the gap is the smallest and this pressure distribution causes a moment on the slab that *tends to misalign* the slab with the surface. These phenomena are readily apparent when pressing or lifting a flat object from a wet or oily surface. In particular, it may be nearly impossible to lift the object straight up; however, once one corner or edge is tilted up, the object is much easier to lift completely.

**Exercise 9.23**. Show that the lubrication approximation can be extended to viscous flow within narrow gaps $h(x,y,t)$ that depend on two spatial coordinates. Start from (4.10) and (9.1), and use Cartesian coordinates oriented so that $x$-$y$ plane is locally tangent to the center-plane of the gap. Scale the equations using a direct extension of (9.14):

$x^* = x/L$ , $y^* = y/L$ , $z^* = z/h = y/\varepsilon L$ , $t^* = Ut/L$ , $u^* = u/U$ , $v^* = v/U$ , $w^* = w/\varepsilon U$ , and $p^* = p/P_a$.

where $L$ is the characteristic distance for the gap thickness to change in either the $x$ or $y$ direction, and $\varepsilon = h/L$. Simplify these equation when $\varepsilon^2 \mathrm{Re}_L \to 0$, but $\mu UL/P_a h^2$ remains of order unity to find:

$$0 \cong -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial z^2} \ , \quad 0 \cong -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\partial^2 v}{\partial z^2} \ , \text{ and } 0 \cong -\frac{1}{\rho}\frac{\partial p}{\partial z}.$$

**Solution 9.23**. In Cartesian coordinates the starting equations (4.10) and the three components of (8.1) are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \,,$$

$$x\text{:} \qquad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right),$$

$$y\text{:} \qquad \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right), \text{ and}$$

$$z\text{:} \qquad \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right).$$

First insert the given scaling into the continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{U}{L}\frac{\partial u^*}{\partial x^*} + \frac{U}{L}\frac{\partial v^*}{\partial y^*} + \frac{\varepsilon U}{h}\frac{\partial w^*}{\partial z^*} = \frac{U}{L}\left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*}\right) = 0 \,.$$

Thus, there are no simplifications of the continuity equation as $\varepsilon^2 \mathrm{Re}_L \to 0$, so it remains unchanged.

Now insert the given scaling into the $x$-momentum equation, to find:

$$\frac{U}{L/U}\frac{\partial u^*}{\partial t^*} + \frac{U^2}{L}u^*\frac{\partial u^*}{\partial x^*} + \frac{U^2}{L}v^*\frac{\partial u^*}{\partial y^*} + \frac{\varepsilon U^2}{h}w^*\frac{\partial u^*}{\partial z^*} = -\frac{P_a}{\rho L}\frac{\partial p^*}{\partial x^*} + \nu\left(\frac{U}{L^2}\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{U}{L^2}\frac{\partial^2 u^*}{\partial y^{*2}} + \frac{U}{h^2}\frac{\partial^2 u^*}{\partial z^{*2}}\right),$$

Simplify using $\varepsilon = h/L$, and collect terms with like coefficients:

$$\frac{U^2}{L}\left(\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} + w^*\frac{\partial u^*}{\partial z^*}\right) = -\frac{P_a}{\rho L}\frac{\partial p^*}{\partial x^*} + \nu\frac{U}{L^2}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right) + \nu\frac{U}{h^2}\frac{\partial^2 u^*}{\partial z^{*2}} \,.$$

Multiply through by $\rho h^2/\mu U$, and use $\varepsilon = h/L$ and $\mathrm{Re}_L = \rho UL/\mu$ to reach:

$$\varepsilon^2 \mathrm{Re}_L\left(\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} + w^*\frac{\partial u^*}{\partial z^*}\right) = -\frac{P_a h^2}{\mu UL}\frac{\partial p^*}{\partial x^*} + \varepsilon^2\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right) + \frac{\partial^2 u^*}{\partial z^{*2}} \,.$$

When $\varepsilon^2 \mathrm{Re}_L \to 0$, but $\mu UL/P_a h^2$ remains of order unity this equation simplifies to:

$$0 \cong -\frac{P_a h^2}{\mu UL}\frac{\partial p^*}{\partial x^*} + \frac{\partial^2 u^*}{\partial z^{*2}}, \text{ or in dimensional form: } 0 \cong -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2} \,.$$

The procedure for the $y$-momentum equation is the same as that for the $x$-momentum equation with $u$ replaced by $v$ and $\partial p/\partial x$ replaced by $\partial p/\partial y$. The final dimensionless and dimensional forms are:

$$0 \cong -\frac{P_a h^2}{\mu U L}\frac{\partial p^*}{\partial y^*} + \frac{\partial^2 v^*}{\partial z^{*2}}, \text{ or in dimensional form: } 0 \cong -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2}.$$

The $z$-momentum equation is different that the other two because the scaling of $w$ differs from that of $u$ and $v$, and the pressure gradient involves $z$ (not $x$ or $y$). Inserting the given scaling leads to:

$$\frac{\varepsilon U}{L/U}\frac{\partial w^*}{\partial t^*} + \frac{\varepsilon U^2}{L}u^*\frac{\partial w^*}{\partial x^*} + \frac{\varepsilon U^2}{L}v^*\frac{\partial w^*}{\partial y^*} + \frac{\varepsilon^2 U^2}{h}w^*\frac{\partial w^*}{\partial z^*} = -\frac{P_a}{\rho L}\frac{\partial p^*}{\partial z^*} + \nu\left(\frac{\varepsilon U}{L^2}\frac{\partial^2 w^*}{\partial x^{*2}} + \frac{\varepsilon U}{L^2}\frac{\partial^2 w^*}{\partial y^{*2}} + \frac{\varepsilon U}{h^2}\frac{\partial^2 w^*}{\partial z^{*2}}\right)$$

As before, simplify using $\varepsilon = h/L$, and collect terms with like coefficients:

$$\frac{\varepsilon U^2}{L}\left(\frac{\partial w^*}{\partial t^*} + u^*\frac{\partial w^*}{\partial x^*} + v^*\frac{\partial w^*}{\partial y^*} + w^*\frac{\partial w^*}{\partial z^*}\right) = -\frac{P_a}{\rho L}\frac{\partial p^*}{\partial z^*} + \nu\frac{\varepsilon U}{L^2}\left(\frac{\partial^2 w^*}{\partial x^{*2}} + \frac{\partial^2 w^*}{\partial y^{*2}}\right) + \nu\frac{\varepsilon U}{h^2}\frac{\partial^2 w^*}{\partial z^{*2}}.$$

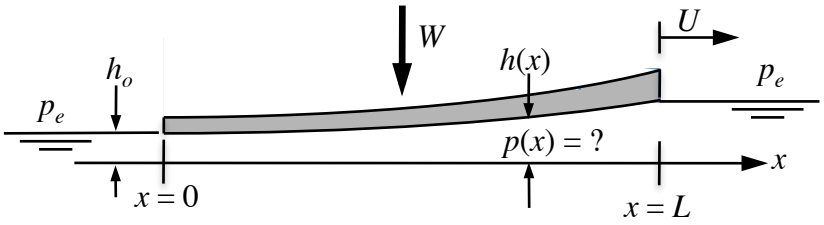
Multiply through by $\rho\varepsilon h^2/\mu U$, and use $\varepsilon = h/L$ and $\mathrm{Re}_L = \rho UL/\mu$ to reach:

$$\varepsilon^4\,\mathrm{Re}_L\left(\frac{\partial w^*}{\partial t^*} + u^*\frac{\partial w^*}{\partial x^*} + v^*\frac{\partial w^*}{\partial y^*} + w^*\frac{\partial w^*}{\partial z^*}\right) = -\frac{P_a h^2}{\mu UL}\frac{\partial p^*}{\partial z^*} + \varepsilon^4\left(\frac{\partial^2 w^*}{\partial x^{*2}} + \frac{\partial^2 w^*}{\partial y^{*2}}\right) + \varepsilon^2\frac{\partial^2 w^*}{\partial z^{*2}}.$$

When $\varepsilon^2\mathrm{Re}_L \to 0$, but $\mu UL/P_a h^2$ remains of order unity this equation simplifies to:

$$0 \cong -\frac{P_a h^2}{\mu UL}\frac{\partial p^*}{\partial z^*}, \text{ or in dimensional form: } 0 \cong -\frac{1}{\rho}\frac{\partial p}{\partial z}.$$

**Exercise 9.24**. A squeegee is pulled across a smooth flat stationary surface at a constant speed $U$. The gap between the squeegee and surface is $h(x) = h_o\exp\{+\alpha x/L\}$ and this gap is filled with a fluid of constant density $\rho$ and viscosity $\mu$. If the squeegee is wetted by the fluid for $0 \le x \le L$, and the pressure in the surrounding air is $p_e$, what is the pressure $p(x)$ distribution under the squeegee and what force $W$ perpendicular to the surface is needed to hold the squeegee in place? Ignore gravity in your work.




**Solution 9.24**. Follow the solution approach in Example 9.3. Here the pressure gradient will be:

$$\frac{dp(x)}{dx} = -\frac{12\mu}{h^3(x)}C_1 - \frac{6\mu U}{h^2(x)} = -\frac{12\mu}{h_o^3}C_1 e^{-3\alpha x/L} - \frac{6\mu U}{h_o^2}e^{-2\alpha x/L} \ .$$

Integrate to find the pressure distribution:

$$p(x) = +\frac{12\mu L}{3h_o^3\alpha}C_1 e^{-3\alpha x/L} + \frac{6\mu UL}{2h_o^2\alpha}e^{-2\alpha x/L} + C_2 = \frac{3\mu UL}{h_o^2\alpha}\left(\frac{4}{3Uh_o}C_1 e^{-3\alpha x/L} + e^{-2\alpha x/L}\right) + C_2 \ .$$

The boundary conditions are $p(0) = p(L) = p_e$, and these lead to two equations for the two constants $C_1$ and $C_2$.

$$p_e = \frac{3\mu UL}{h_o^2\alpha}\left(\frac{4}{3Uh_o}C_1 + 1\right) + C_2 \ , \text{ and } \ p_e = \frac{3\mu UL}{h_o^2\alpha}\left(\frac{4}{3Uh_o}C_1 e^{-3\alpha} + e^{-2\alpha}\right) + C_2 \ . \qquad (1,2)$$

Subtract (2) from (1) to find $C_1$:

$$0 = \frac{3\mu UL}{h_o^2\alpha}\left(\frac{4}{3Uh_o}C_1(1 - e^{-3\alpha}) + 1 - e^{-2\alpha}\right) \ \text{ or } \ C_1 = -\frac{3Uh_o}{4}\frac{1 - e^{-2\alpha}}{1 - e^{-3\alpha}} \ .$$

Substitute this into (1) to find $C_2$:

$$C_2 = p_e - \frac{3\mu UL}{h_o^2\alpha}\left(\frac{e^{-2\alpha} - e^{-3\alpha}}{1 - e^{-3\alpha}}\right).$$

Use these constants to construct $p(x)$:

$$p(x) - p_e = \frac{3\mu UL}{h_o^2\alpha}\left(-\frac{1 - e^{-2\alpha}}{1 - e^{-3\alpha}}e^{-3\alpha x/L} + e^{-2\alpha x/L} - \frac{e^{-2\alpha} - e^{-3\alpha}}{1 - e^{-3\alpha}}\right) \ .$$

The force (or load) $W$ is determined from the integral:

$$W = \int_0^L \big(p(x) - p_e\big)dx = \frac{3\mu LU}{h_o^2\alpha}\int_0^L \left(-\frac{1 - e^{-2\alpha}}{1 - e^{-3\alpha}}e^{-3\alpha x/L} + e^{-2\alpha x/L} - \frac{e^{-2\alpha} - e^{-3\alpha}}{1 - e^{-3\alpha}}\right)dx \ .$$

The integration of the exponentials may be done directly:

$$W = \frac{3\mu LU}{h_o^2\alpha}\left(-\frac{1 - e^{-2\alpha}}{1 - e^{-3\alpha}}\left[\frac{e^{-3\alpha x/L}}{-3\alpha/L}\right]_0^L + \left[\frac{e^{-2\alpha x/L}}{-2\alpha/L}\right]_0^L - \frac{e^{-2\alpha} - e^{-3\alpha}}{1 - e^{-3\alpha}}\big[x\big]_0^L\right).$$

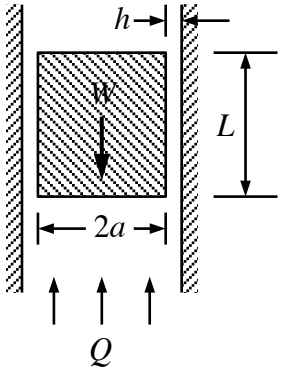
Continue with the evaluation:

$$W = \frac{3\mu LU}{h_o^2\alpha}\left(\frac{L}{3\alpha}\frac{1-e^{-2\alpha}}{1-e^{-3\alpha}}\left(e^{-3\alpha}-1\right)-\frac{L}{2\alpha}\left(e^{-2\alpha}-1\right)-\frac{e^{-2\alpha}-e^{-3\alpha}}{1-e^{-3\alpha}}L\right)$$
$$= \frac{3\mu LU}{h_o^2\alpha}\left(\frac{L}{6\alpha}\left(1-e^{-2\alpha}\right)-\frac{e^{-2\alpha}-e^{-3\alpha}}{1-e^{-3\alpha}}L\right)$$

And, when $\alpha << 1$, this reduces to $W = \frac{\alpha\mu L^2 U}{2h_o^2}$, which is the same as the result of Example 9.3.

This occurs because the two shapes become the same; $h(x) = h_o\exp\{+\alpha x/L\} \approx h_o(1 + \alpha x/L)$ when $\alpha << 1$.

**Exercise 9.25**. A close-fitting solid cylinder with net weight $W$ (= actual weight – buoyancy), length $L$, and radius $a$ is centered in and may slide along the axis of a long vertical tube with radius $a + h$, where $h << a$. The tube is filled with oil having constant viscosity $\mu$ that is pumped slowly upward at a volume flow rate $Q$.
a) Use dimensional analysis to find a scaling law for the value of $Q$ that holds the cylinder stationary when fluid inertia is unimportant.
b) Use the lubrication approximation and assume that the pressure is uniform above and below the cylinder to determine a formula for the value of $Q$ that holds the cylinder stationary.

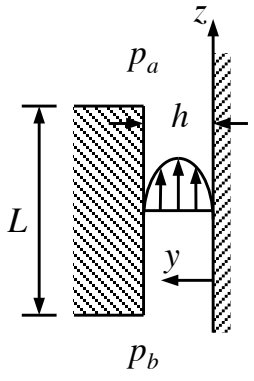



**Solution 9.25**. a) There are six parameters ($Q$, $W$, $L$, $a$, $h$, $\mu$) so the units matrix (which has rank 3) is:

| | $Q$ | $W$ | $L$ | $a$ | $h$ | $\mu$. |
|---|---|---|---|---|---|---|
| M | 0 | 1 | 0 | 0 | 0 | 1 |
| L | 3 | 1 | 1 | 1 | 1 | -1 |
| T | -1 | -2 | 0 | 0 | 0 | -1 |

The 6 – 3 = 3 dimensionless groups are: $\Pi_1 = Q\mu/LW$, $\Pi_2 = a/L$, and $\Pi_3 = h/L$, so the scaling law is: $Q\mu/LW = f(a/L, h/L)$, where $f$ is an undetermined function.

b) Under the lubrication approximation, the flow in the narrow gap will be parabolic. Choose the $z$-axis vertically upward, and let $y$ be a coordinate that is zero on the tube wall and that increases toward the tube centerline as shown in the drawing to the left. Both the tube and the cylinder are stationary so $w(y = 0) = w(y = h) = 0$. With these coordinates, the fluid velocity in the gap will be:




$$w(y) = -\frac{h^2}{2\mu}\left(\frac{dp}{dz}\right)\frac{y}{h}\left(1-\frac{y}{h}\right) = -\frac{h^2}{2\mu}\left(\frac{p_a - p_b}{L}\right)\frac{y}{h}\left(1-\frac{y}{h}\right)$$

where $p_a$ and $p_b$ are the (uniform) pressures in the fluid above and below the cylinder, respectively. The cylinder's net weight $W$ will be supported by shear stress on its sides [area = $2\pi aL$] and the pressure difference between its ends acting on its round surfaces:

$$W = 2\pi aL\mu\left(-\frac{dw}{dy}\right)_{y=h} + \left(p_b - p_a\right)\pi a^2 = 2\pi aL\mu\left(-\frac{h}{2\mu}\left(\frac{p_a - p_b}{L}\right)\right) + \left(p_b - p_a\right)\pi a^2,$$

or: $W = \left(p_b - p_a\right)\left[\pi ah + \pi a^2\right]$. In addition, all the oil flow must flow past the cylinder. The gap-center circle has circumference $2\pi(h + a/2)$, so

$$Q = 2\pi\left(a+\tfrac{1}{2}h\right)\int_0^h w(y)\,dy = 2\pi\left(a+\tfrac{1}{2}h\right)\left(-\frac{h^3}{12\mu}\left(\frac{p_a - p_b}{L}\right)\right).$$

Eliminate the pressure difference from last two equations for $Q$ and $W$ to find:

$$Q = \frac{W\left(a+\tfrac{1}{2}h\right)h^3}{6\mu L\left[ah+a^2\right]} \cong \frac{Wh^3}{6\mu aL},$$

where the second more approximate form is valid as $h/a \to 0$.

**Exercise 9.26**. A thin film of viscous fluid is bounded below by a flat stationary plate at $z = 0$. If the in-plane velocity at the upper film surface, $z = h(x,y,t)$, is $\mathbf{U} = U(x,y,t)\mathbf{e}_x + V(x,y,t)\mathbf{e}_y$, use the equations derived in Exercise 8.19 to produce the Reynolds equation for constant-density thin-flim lubrication:

$$\nabla\cdot\left[\left(\frac{h^3}{\mu}\right)\nabla p\right] = 12\frac{\partial h}{\partial t} + 6\nabla\cdot(h\mathbf{U})$$

where $\nabla = \mathbf{e}_x(\partial/\partial x) + \mathbf{e}_y(\partial/\partial y)$ merely involves the two in-plane dimensions.

**Solution 9.26**. For the specified geometry, solutions for the $x$- and $y$-direction equations are given by appropriately evaluated versions of (8.19):

$$u \cong -\frac{h^2}{2\mu}\frac{\partial p}{\partial x}\frac{z}{h}\left(1-\frac{z}{h}\right) + U\frac{z}{h} \text{ , and } v \cong -\frac{h^2}{2\mu}\frac{\partial p}{\partial y}\frac{z}{h}\left(1-\frac{z}{h}\right) + V\frac{z}{h}.$$

where $h$, $U$, and $V$ are functions of $x$, $y$, and $t$.

Now consider a stationary nearly-rectangular control volume of with elemental base area $dxdy$ and height $h(x,y,t)$. If the flat vertical sides of the CV are perpendicular to the coordinate directions, then an integral statement of conservation of mass will include one term involving the time-variation of $h$ and four flux terms, one for each vertical side:

$$\rho\frac{\partial h}{\partial t}dxdy - \rho dy\int_0^{h(x,y,t)} u dz - \rho dx\int_0^{h(x,y,t)} v dz + \rho dy\int_0^{h(x+dx,y,t)} u dz + \rho dx\int_0^{h(x,y+dy,t)} v dz = 0.$$

For constant density, this can be simplified and rearranged:

$$\frac{\partial h}{\partial t} + \frac{1}{dx}\left[\int_0^{h(x+dx,y,t)} u dz - \int_0^{h(x,y,t)} u dz\right] + \frac{1}{dy}\left[\int_0^{h(x,y+dy,t)} v dz - \int_0^{h(x,,t)} v dz\right] = 0, \text{ or } \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left[\int_0^{h(x,y,t)} u dz\right] + \frac{\partial}{\partial y}\left[\int_0^{h(x,,t)} v dz\right] = 0.$$

The two integrations in [,]-brackets in the second equality can be completed using the equations above for $u$ and $v$.

$$\int_0^h u dz \cong -\frac{h^2}{2\mu}\frac{\partial p}{\partial x}\int_0^h \frac{z}{h}\left(1-\frac{z}{h}\right)dz + U\int_0^h \frac{z}{h}dz$$
$$= -\frac{h^3}{2\mu}\frac{\partial p}{\partial x}\int_0^1 \beta(1-\beta)d\beta + Uh\int_0^1 \beta d\beta$$
$$= -\frac{h^3}{2\mu}\frac{\partial p}{\partial x}\left(\frac{1}{2}-\frac{1}{3}\right) + Uh\left(\frac{1}{2}\right)$$
$$= -\frac{h^3}{12\mu}\frac{\partial p}{\partial x} + \frac{Uh}{2},$$

, and

$$\int_0^h v dz \cong -\frac{h^2}{2\mu}\frac{\partial p}{\partial y}\int_0^h \frac{z}{h}\left(1-\frac{z}{h}\right)dz + V\int_0^h \frac{z}{h}dz$$
$$= -\frac{h^3}{2\mu}\frac{\partial p}{\partial y}\int_0^1 \beta(1-\beta)d\beta + Vh\int_0^1 \beta d\beta$$
$$= -\frac{h^3}{2\mu}\frac{\partial p}{\partial y}\left(\frac{1}{2}-\frac{1}{3}\right) + Vh\left(\frac{1}{2}\right)$$
$$= -\frac{h^3}{12\mu}\frac{\partial p}{\partial y} + \frac{Vh}{2}.$$

The result of the continuity equation becomes:

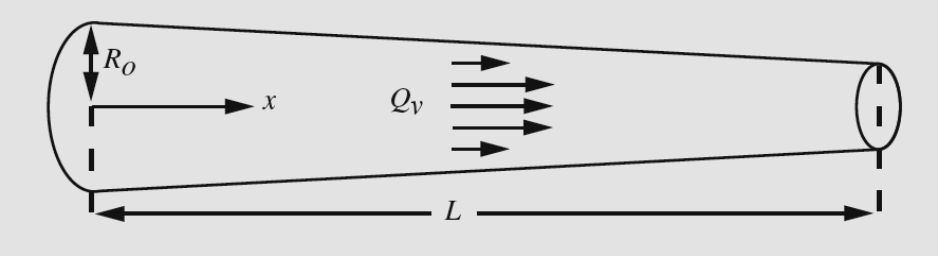
$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(-\frac{h^3}{12\mu}\frac{\partial p}{\partial x} + \frac{Uh}{2}\right) + \frac{\partial}{\partial y}\left(-\frac{h^3}{12\mu}\frac{\partial p}{\partial y} + \frac{Vh}{2}\right) = 0.$$

Rearrange this equation and use $\nabla = \mathbf{e}_x(\partial/\partial x) + \mathbf{e}_y(\partial/\partial y)$ and $\mathbf{U} = U\mathbf{e}_x + V\mathbf{e}_y$,

$$12\frac{\partial h}{\partial t} + 6\frac{\partial}{\partial x}(Uh) + 6\frac{\partial}{\partial y}(Vh) = \frac{\partial}{\partial x}\left(\frac{h^3}{\mu}\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{h^3}{\mu}\frac{\partial p}{\partial y}\right) \text{ or } 12\frac{\partial h}{\partial t} + 6\nabla\cdot(h\mathbf{U}) = \nabla\cdot\left(\frac{h^3}{\mu}\nabla p\right).$$

Here it is worth noting that when $U$, $V$, and $h$ are known, the solution of final equation specifies the pressure in the fluid film.

**Exercise 8.21**. Fluid of density $\rho$ and viscosity $\mu$ flows inside a long tapered tube of length $L$ and radius $R(x) = (1 - \alpha x/L)R_o$ where $\alpha < 1$, and $R_o << L$.
a) Estimate the volume discharge rate $Q_V$ through the tube, for a given pressure difference $\Delta p$ sustained between the inlet and the outlet.
b) Discuss the range of validity of your solution in terms of the parameters of the problem.




**Solution 8.21**. a) If the tube diameter changes slowly enough, the flow will be in equilibrium with the local diameter of the pipe and the axial velocity profile will be the same as that in an untapered pipe of the same diameter. For flow in an untapered pipe: $Q_v = U_{ave}A$, and $U_{ave} = -\dfrac{d^2}{32\mu}\dfrac{dp}{dx}$. For this problem let $U_{ave} = U_{ave}(x)$ since $A = A(x)$ and $Q_v = U_{ave}(x)A(x) = const$, therefore: $U_{ave}(x) = -\dfrac{R^2(x)}{8\mu}\dfrac{dp}{dx}$. Eliminate $U_{ave}(x)$ from the last two equations and use $A(x) = \pi R^2(x)$ to get:

$$Q_v = -\frac{R^2(x)}{8\mu}\frac{dp}{dx}A(x) = -\frac{\pi R^4(x)}{8\mu}\frac{dp}{dx} \tag{1}$$

Now solve for $-dp/dx$ and integrate in $x$ from 0 to $L$:

$$-\big(p(L) - p(0)\big) = \int_0^L \frac{8\mu Q_v}{\pi R^4(x)}dx = \frac{8\mu Q_v}{\pi R_o^4}\int_0^L \left(1 - \alpha\frac{x}{L}\right)^{-4} dx = \frac{8\mu Q_v L}{\pi R_o^4 \alpha}\int_1^{1-\alpha} \zeta^{-4} d\zeta$$

The integral is elementary and is easily evaluated, yielding:

$$p(0) - p(L) = \frac{8\mu Q_v L}{3\pi R_o^4 \alpha}\left(\frac{1}{(1-\alpha)^3} - 1\right).$$

Inverting this to find $Q_v$ produces:

$$Q_v = \frac{3\pi R_o^4 \alpha}{8\mu L}\big(p(0) - p(L)\big)\left(\frac{1}{(1-\alpha)^3} - 1\right)^{-1}.$$

This result should be expanded for $\alpha << 1$,

$$\left(\frac{1}{(1-\alpha)^3} - 1\right)^{-1} = \frac{(1-\alpha)^3}{1-(1-\alpha)^3} = \frac{1 - 3\alpha + 3\alpha^2 - \alpha^3}{3\alpha - 3\alpha^2 + \alpha^3} = \frac{1}{3\alpha}\left(\frac{1 - 3\alpha + ...}{1 - \alpha + ...}\right) \approx \frac{1}{3\alpha}(1 - 3\alpha)(1 + \alpha) \approx \frac{1 - 2\alpha}{3\alpha}.$$

Therefore:

$$Q_v = \frac{\pi R_o^4}{8\mu}\left(\frac{p(0) - p(L)}{L}\right)(1 - 2\alpha),$$

so for a fixed pressure difference $Q_v$ is larger when the pipe expands ($\alpha < 0$) and is smaller when the pipe contracts ($\alpha > 0$).

b) The flow accelerates or decelerates inside a contracting or expanding pipe, respectively. The above solution will only be valid if fluid acceleration can be ignored compared to $\frac{1}{\rho}\frac{dp}{dx}$ or $\nu\nabla^2 u$ (recall that these two terms are set equal to obtain the solution for the velocity profile). Hence we need only consider one of them, and therefore must have: $\left|u\frac{\partial u}{\partial x}\right| << \left|\frac{1}{\rho}\frac{dp}{dx}\right|$ where $u\frac{\partial u}{\partial x}$ is a typical acceleration term. This requirement implies:

$$\left|U_{ave}\frac{\partial U_{ave}}{\partial x}\right| = \left|\frac{R^2(x)}{8\mu}\frac{dp}{dx}\frac{\partial}{\partial x}\left(\frac{Q_v}{\pi R^2(x)}\right)\right| << \left|\frac{1}{\rho}\frac{dp}{dx}\right|$$

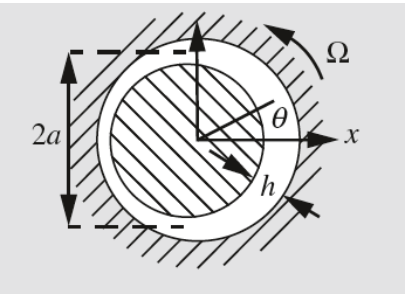
where the two relationships $U_{ave} = -\frac{R^2(x)}{8\mu}\frac{dp}{dx} = \frac{Q_v}{\pi R^2(x)}$ has been used outside and inside the derivative. Cancel common factors, evaluate the derivative using $R(x) = (1 - \alpha x/L)R_o$ and assume $R(x) \approx R_o$ to find:

$$\left|\frac{R^2(x)Q_v}{4\nu\pi R^3(x)}\left(\frac{\alpha R_o}{L}\right)\right| \approx \frac{Q_v\alpha}{4\nu\pi L} << 1 \ .$$

When this is true, the solution for part a) should be OK.

**Exercise 9.28**. A circular lubricated bearing of radius $a$ holds a stationary round shaft. The bearing hub rotates at angular rate $\Omega$ as shown. A load per unit depth on the shaft, **W**, causes the center of the shaft to be displaced from the center of the rotating hub by a distance $\varepsilon h_o$, where $h_o$ is the average gap thickness and $h_o << a$. The gap is filled with an incompressible oil of viscosity $\mu$. Neglect the shear stress contribution to **W**.

a) Determine a dimensionless scaling law for |**W**|.
b) Determine **W** by assuming a lubrication flow profile in the gap and $h(\theta) = h_o(1 + \varepsilon\cos\theta)$ with $\varepsilon \ll 1$.
c) If **W** is increased a little bit, is the lubrication action stabilizing?




**Solution 9.28**. a) Follow the usual dimensional analysis approach. The flow is caused by the balance of pressure and viscous forces, so the fluid inertia (i.e. $\rho$) doesn't matter.

• Boundary condition & material parameters:
$\mu$ = viscosity of the fluid
$\Omega$ = rotation speed of the shaft
$h_o$ = mean gap separation
$a$ = radius of the bearing
$\varepsilon$ = accentricity parameter

Solution parameter: W = |**W**| = magnitude of the applied load per unit depth

• Create the parameter matrix:

| | $W$ | $\Omega$ | $h_o$ | $\mu$ | $a$ | $\varepsilon$ |
|---|---|---|---|---|---|---|
| Mass: | 1 | 0 | 0 | 1 | 0 | 0 |
| Length: | 0 | 0 | 1 | -1 | 1 | 0 |
| Time: | -2 | -1 | 0 | -1 | 0 | 0 |

This matrix has rank three.

• Determine the number of dimensionless groups: 6 parameters - 3 dimensions = 3 groups

• Construct the dimensionless group: $\Pi_1 = \dfrac{W}{\mu\Omega a}$, $\Pi_2 = \dfrac{h_o}{a}$, and $\Pi_3 = \varepsilon$.

• Write a dimensionless law: $\dfrac{W}{\mu\Omega a} = fn\left(\dfrac{h_o}{a}, \varepsilon\right)$.

b) Because the fluid gap is very narrow, consider a new coordinate: $z$ that is zero on the surface of the shaft and equal to $h(\theta)$ at the outer rotating bearing surface, and $\xi = a\theta$ = downstream distance. Place a stationary control volume in the gap that encloses the fluid in the gap between angles $\theta_1$ and $\theta_2$. Conservation of mass for the gap-flow will imply that:

$$\left[\int_0^h u(z)dz\right]_{\theta=\theta_1} = \left[\int_0^h u(z)dz\right]_{\theta=\theta_2} = C_1 = const.$$

(Note, this situation is subtly different than the sloped bearing worked in lecture; in the unwrapped geometry, the upper wall is moving but its geometric shape does not move.) Conservation of momentum in the angular direction in the gap will be satisfied locally by using the Couette flow solution for $u(z)$: i.e.

$$C_1 = \int_0^h \left(-\frac{z(h-z)}{2\mu a}\frac{dP}{d\theta} + \frac{\Omega a z}{h}\right)dz = -\frac{h^3}{12\mu a}\frac{dP}{d\theta} + \frac{\Omega a h}{2}$$

Treat this result as a differential equation for $P(\theta)$:

$$\frac{dP}{d\theta} = \frac{-12\mu a C_1 + 6\mu\Omega a^2 h}{h^3} = \frac{-12\mu a C_1 + 6\mu\Omega a^2 h_o(1+\varepsilon\cos\theta)}{h_o^3(1+\varepsilon\cos\theta)^3}$$

Even though integration of this equation can be completed exactly, it's tedious and unnecessary because $\varepsilon << 1$. Therefore, expand for $\varepsilon << 1$ and only keep the linear terms in $\varepsilon$:

$$\frac{dP}{d\theta} = \frac{-12\mu a C_1 + 6\mu\Omega a^2 h_o}{h_o^3} + \frac{36\mu a C_1 - 12\mu\Omega a^2 h_o}{h_o^3}\varepsilon\cos\theta + \ldots$$

Integrate from 0 to $\theta$:

$$P(\theta) - P(0) = \frac{-12\mu a C_1 + 6\mu\Omega a^2 h_o}{h_o^3}\theta + \frac{36\mu a C_1 - 12\mu\Omega a^2 h_o}{h_o^3}\varepsilon\sin\theta + \ldots$$

For the pressure to be single valued [i.e. $P(2\pi) = P(0)$], we must have $C_1 = \Omega a h_o/2$ (this makes the linear-in-$\theta$ term in the last equation equal to zero). Therefore $P(\theta) - P(0) = 6\mu\Omega a^2\varepsilon\sin\theta/h_o^2$.

Now use this pressure distribution to determine the load on the shaft and ignore any net contributions from shear stresses. The external load on the shaft, which is equal an opposite to the fluid load applied to the shaft, may have both $x$- and $y$-components, so

$$\mathbf{W} = (W_x, W_y) = -\int_0^{2\pi}\left(P(0) - P(\theta)\right)\vec{n}a\,d\theta,$$

where $\mathbf{n} = (\cos\theta, \sin\theta)$ is the outward normal from the shaft's surface.

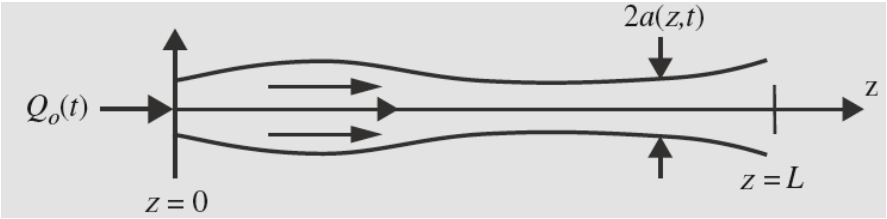
$$\mathbf{W} = \frac{6\mu\Omega a^3\varepsilon}{h_o^2}\int_0^{2\pi}\sin\theta\left(\cos\theta, \sin\theta\right)d\theta = \left(0, +\frac{6\pi\mu\Omega a^3\varepsilon}{h_o^2}\right)$$

This answer matches the dimensional analysis result. However, the external load applied to the shaft pushes the shaft up! This seemingly odd result, makes sense if you think about what's going on. For the fluid to pass through the narrowest part of the gap at $\theta = \pi$ where the viscous forces are largest, there must be a favorable (i.e negative) d$P$/d$\theta$ in this region. This means that the pressure above the shaft (near $\theta = \pi/2$) must be greater than the pressure below the shaft (near $\theta = 3\pi/2$). With high pressure above and low pressure below, the lubricant must be trying to push the shaft down. In order to maintain shaft position, there must be an upward-directed externally-applied load on the shaft.

c) If $W_y$ is impulsively increased by a small amount, the shaft will move vertically. The lubrication action will then push the shaft to the left. Eventually a new steady state will be reached with the shaft misaligned in the bearing (shaft center to the left) with a larger value of $\varepsilon$ than the original configuration. So, the lubrication action can be considered stabilizing.

**Exercise 9.29**. As a simple model of small-artery blood flow, consider slowly varying viscous flow through a round flexible tube with inlet at $z = 0$ and outlet at $z = L$. At $z = 0$, the volume flux entering the tube is $Q_o(t)$. At $z = L$, the pressure equals the exterior pressure $p_e$. The radius of the tube, $a(z,t)$, expands and contracts in proportion to pressure variations within the tube so that (*i*) $a - a_e = \gamma(p - p_e)$, where $a_e$ is the tube radius when the pressure, $p(z,t)$, in the tube is equal to $p_e$, and $\gamma$ is a positive constant. Assume the local volume flux, $Q(z,t)$, is related to $\partial p/\partial z$ by (*ii*) $Q = -\left(\pi a^4/8\mu\right)\left(\partial p/\partial z\right)$.

a) By conserving mass, find a partial differential equation that relates $Q$ and $a$.
b) Combine (*i*), (*ii*), and the result of part a) into one partial differential equation for $a(z,t)$.
c) Determine $a(z)$ when $Q_o$ is a constant and the flow is perfectly steady.




**Solution 9.29**. a) Conserve mass in a differential disk control volume perpendicular to the flow direction that has circular inlet and outlet surfaces located at $z$ and $z + \Delta z$ with radii of $a(z,t)$ and $a(z + \Delta z,t)$. There is no flow through the tube wall so:

$$\frac{\partial}{\partial t}\int_{CV}\rho dV - \int_0^{a(z,t)}\rho u(r,t)2\pi r dr + \int_0^{a(z+\Delta z,t)}\rho u(r,t)2\pi r dr = 0\,.$$

Noting that $Q(z,t) = \int_{r=0}^{r=R(z,t)} u(r,t)2\pi r dr$ and that the CV volume is $\pi a^2(z,t)\Delta z$ allows the equation above to be simplified to:

$$\left(\partial/\partial t\right)\left(\pi a^2\right) + \left(Q(z+\Delta z,t) - Q(z,t)\right)/\Delta z = 0\,,$$

or taking the limit as $\Delta z \to 0$: $\dfrac{\partial}{\partial t}\left(\pi a^2\right) + \dfrac{\partial Q}{\partial z} = 0$.

b) Differentiate (*i*) to find: $\partial a/\partial z = \gamma\,\partial p/\partial z$. Put this into (*ii*) to get: $Q = -\left(\pi a^4/8\gamma\mu\right)\left(\partial a/\partial z\right)$. Insert this into the result of part a): $\dfrac{\partial}{\partial t}\left(a^2\right) = \dfrac{\partial}{\partial z}\left(\dfrac{a^4}{8\gamma\mu}\dfrac{\partial a}{\partial z}\right)$.

c) For steady flow, the part a) result is not needed. From part b), (*i*) and (*ii*) imply: $Q_o = -\left(\pi a^4/8\gamma\mu\right)\left(\partial a/\partial z\right)$. Integrate once to find: $\dfrac{\pi a^5}{40\gamma\mu} = -Q_o z + C$ where $C$ is a constant. At $z = L$, $a = a_e$ so $C = \dfrac{\pi a_e^5}{40\gamma\mu} + Q_o L$. Thus the final answer is: $a(z) = \left[\dfrac{40\gamma\mu Q_o}{\pi}(L - z) + a_e^5\right]^{1/5}$.

**Exercise 9.30**. Consider a simple model of flow from a tube of toothpaste. A liquid with viscosity $\mu$ and density $\rho$ is squeezed out of a round horizontal tube having radius $a(t)$. In your work, assume that $a$ is decreasing and use cylindrical coordinates with the $z$-axis coincident with the centerline of the tube. The tube is closed at $z = 0$, but is open to the atmosphere at $z = L$. Ignore gravity.

a) If $w$ is the fluid velocity along the $z$-axis, show that: $za\dfrac{da}{dt} + \int_0^a w(z,R,t)RdR = 0$.

b) Determine the pressure distribution, $p(z) - p(L)$, by assuming the flow in the tube can be treated within the lubrication approximation by setting $w(z,R,t) = -\dfrac{1}{4\mu}\dfrac{dp}{dz}\left(a^2(t) - R^2\right)$.

c) Find the cross-section-average flow velocity $w_{ave}(z,t)$ in terms of $z$, $a$, and $da/dt$.

d) If the pressure difference between $z = 0$ and $z = L$ is $\Delta P$, what is the volume flux exiting the tube as a function of time. Does this answer partially explain why fully emptying a toothpaste tube by squeezing it is essentially impossible?

**Solution 9.30**. a) Choose a CV that encloses the fluid inside the tube from the capped end to an axial distance $z$. Conservation of mass implies:

$$\frac{d}{dt}\left(\rho\pi a^2 z\right) + \int_0^a \rho w(z,R,t)2\pi RdR = 0 \quad \text{or} \quad za\frac{da}{dt} + \int_0^a w(z,R,t)RdR = 0.$$

b) Here $p = p(z)$ alone; insert the given velocity profile into the result of part a) and integrate:

$$za\frac{da}{dt} = \frac{1}{4\mu}\frac{dp}{dz}\int_0^a \left(a^2(t) - R^2\right)RdR = \frac{a^4(t)}{16\mu}\frac{dp}{dz}.$$

The ends of this equality form a differential equation for the pressure; integrate from $z$ to $L$:

$$p(z) - p(L) = -\left(8\mu/a^3\right)(da/dt)\left(L^2 - z^2\right).$$

c) Multiply the result of part a) by $2a^{-2}$, insert factors of $\pi$, and recognize the definition of $w_{ave}$:

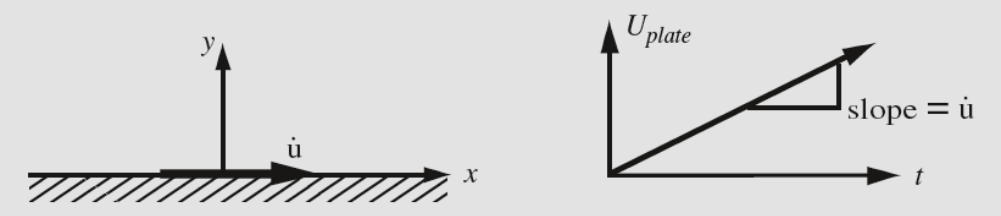
$$\frac{2}{a^2}\int_0^a w(z,R,t)RdR = \frac{1}{\pi a^2}\int_0^a w(z,R,t)2\pi RdR = w_{ave}(z,t) = -\left(\frac{2}{a^2}\right)za\frac{da}{dt} = -\frac{2z}{a}\frac{da}{dt},$$

d) Use $Q(t) = \pi a^2 w_{ave}(L,t)$, and combine the results of parts b) and c) to eliminate $da/dt$. From part b): $p(0) - p(L) = \Delta P = -\dfrac{8\mu}{a^3}\left(\dfrac{da}{dt}\right)\left(L^2 - 0^2\right)$, or $\left(\dfrac{da}{dt}\right) = -\dfrac{a^3\Delta P}{8\mu L^2}$; so

$$Q(t) = \pi a^2 w_{ave}(L,t) = -\pi a^2\frac{2L}{a}\frac{da}{dt} = -2\pi La\frac{da}{dt} = -2\pi La\left(\frac{-a^3\Delta P}{8\mu L^2}\right) = \frac{\pi}{4}\frac{a^4\Delta P}{\mu L}$$

Yes, this answer explains a lot about flow from a toothpaste tube. Here, it is clear that for a finite $\Delta P$ (the pressure one can exert by squeezing), the volume flux from the tube will be proportional to $a^4$. Thus, as the tube empties and its effective cross section decreases, the flow rate from the tube approaches zero. The only hope for better emptying the tube is to decrease $L$, and it is an easily observed fact that more toothpaste can be extracted from a nearly-empty tube by rolling it up from the sealed end.

**Exercise 9.31**. A large flat plate below an infinite stationary incompressible viscous fluid is set in motion with a constant acceleration, $\dot{u}$, at $t = 0$. A prediction for the subsequent fluid motion, $u(y,t)$, is sought.
a) Use dimensional analysis to write a physical law for $u(y,t)$ in this flow.
b) Starting from the $x$-component of (8.1) determine a linear partial differential equation for $u(y,t)$.
c) The linearity of the equation obtained for part c) suggests that $u(y,t)$ must be directly proportional to $\dot{u}$. Simplify your dimensional analysis to incorporate this requirement.
d) Let $\eta = y/(\nu t)^{1/2}$ be the independent variable, and derive a second-order ordinary linear differential equation for the unknown function $f(\eta)$ left from the dimensional analysis.
e) From an analogy between fluid acceleration in this problem and fluid velocity in Stokes' first problem, deduce the solution $u(y,t) = \dot{u}\int_0^t \left[1 - erf\left(y/2\sqrt{\nu t'}\right)\right]dt'$ and show that it solves the equation of part b).
f) Determine $f(\eta)$ and – if your patience holds out – show that it solves the equation found in part d).
g) Sketch the expected velocity profile shapes for several different times. Note the direction of increasing time on your sketch.




**Solution 9.31**. a) The problem parameters are: $\dot{u}$, $\rho$, $y$, $t$, and $\mu$. The solution parameter is u.
• Create the parameter matrix:

| | $u$ | $\dot{u}$ | $\rho$ | $y$ | $t$ | $\mu$ |
|---|---|---|---|---|---|---|
| Mass: | 0 | 0 | 1 | 0 | 0 | 1 |
| Length: | 1 | 1 | -3 | 1 | 0 | -1 |
| Time: | -1 | -2 | 0 | 0 | 1 | -1 |

This matrix has rank three.
• Determine the number of dimensionless groups: 6 parameters - 3 dimensions = 3 groups
• Construct the dimensionless groups: $\prod_1 = u/\dot{u}t$, $\prod_2 = y/(\nu t)^{1/2}$, $\prod_3 = \rho\dot{u}ty/\mu$
• Write a dimensionless law: $u = \dot{u}tF(y/(\nu t)^{1/2}, \rho\dot{u}ty/\mu)$, where $F$ is an unknown function.
b) Assume that $u = u(y,t)$ only!, then $\partial u/\partial x = \partial^2 u/\partial x^2 = 0$. In this case the continuity equation requires $\partial v/\partial y = 0$, so since $v = 0$ at the surface of the plate it is zero everywhere. If the plate is large, there will be no end effects so the pressure gradient must also be zero. This eliminates the nonlinear terms in the equation, the pressure gradient, and one of the viscous terms. The final result being: $\dfrac{\partial u}{\partial t} = \nu\dfrac{\partial^2 u}{\partial y^2}$.
c) If the form of the solution must be linear with respect to $\dot{u}$, then $\prod_3$ can not be a parameter since $\prod_1$ takes care of any linear relationship between $u$ and $\dot{u}$. This means: $u = \dot{u}tf(y/(\nu t)^{1/2})$

d) First convert derivatives, then differentiate being sure to account for the pre-factor of $\dot{u}t$.

$$\frac{\partial}{\partial t}=\frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta}=-\frac{1}{2}\frac{y}{\sqrt{\nu t^3}}\frac{\partial}{\partial \eta}=-\frac{\eta}{2t}\frac{\partial}{\partial \eta},\ \text{and}\ \ \frac{\partial}{\partial y}=\frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta}=\frac{1}{\sqrt{\nu t}}\frac{\partial}{\partial \eta}.$$

Therefore: $\dfrac{\partial u}{\partial t}=\dfrac{\partial}{\partial t}\left[\dot{u}tf(\eta)\right]=\dot{u}f-\dot{u}t\dfrac{\eta}{2t}\dfrac{df}{d\eta}$, and $\nu\dfrac{\partial^2 u}{\partial y^2}=\dfrac{\partial^2}{\partial y^2}\left[\dot{u}tf(\eta)\right]=\nu\dot{u}t\dfrac{1}{\nu t}\dfrac{d^2 f}{d\eta^2}$.

Put these converted derivatives together and cancel common factors, to find:

$$\frac{d^2 f}{d\eta^2}+\frac{\eta}{2}\frac{df}{d\eta}-f=0.$$

e) In Stokes' first problem the plate's velocity was impulsively raised from 0 to $U$ and the velocity field equation and its solution are:

$$\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial y^2}\ \text{and}\ \ u(y,t)=U\left[1-\text{erf}\left(y/2\sqrt{\nu t}\right)\right].$$

In the current problem, the plate's acceleration is impulsively raised from 0 to $\dot{u}$, and the horizontal acceleration of fluid particles is $a(y,t)=\dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}=\dfrac{\partial u}{\partial t}+0$, because $\partial u/\partial x=0$. Thus, by time differentiating the field equation for $u$, the field equation for $a$ and its analogous solution is obtained:

$$\frac{\partial a}{\partial t}=\nu\frac{\partial^2 a}{\partial y^2}\ \text{and}\ \ a(y,t)=\dot{u}\left[1-\text{erf}\left(y/2\sqrt{\nu t}\right)\right].$$

However, $a=\partial u/\partial t$ implies $u(y,t)=\int_{t'=0}^{t'=t}a(y,t')dt'$, so the above analogy suggests the velocity field for the impulsively accelerated plate will be:

$$u(y,t)=\dot{u}\int_{t'=0}^{t'=t}\left[1-\text{erf}\left(y/2\sqrt{\nu t'}\right)\right]dt'.$$

To show that this solution satisfies the field equation for $u$, determine the derivatives using the definition of the error function, $\text{erf}(\zeta)=\left(2/\sqrt{\pi}\right)\int_0^{\zeta}\exp\left(-\xi^2\right)d\xi$.

$$\frac{\partial u}{\partial t}=\dot{u}\frac{\partial}{\partial t}\int_0^t\left[1-\text{erf}\left(y/2\sqrt{\nu t'}\right)\right]dt'=\dot{u}\left[1-\text{erf}\left(y/2\sqrt{\nu t}\right)\right],$$

$$\frac{\partial u}{\partial y}=\dot{u}\frac{\partial}{\partial y}\int_0^t\left[1-\text{erf}\left(y/2\sqrt{\nu t'}\right)\right]dt'=\dot{u}\frac{\partial}{\partial y}\left\{t-\frac{2}{\sqrt{\pi}}\int_0^t\int_0^{y/2\sqrt{\nu t'}}\exp\left(-\xi^2\right)d\xi dt'\right\}$$

$$=\dot{u}\left\{0-\frac{2}{\sqrt{\pi}}\int_0^t\exp\left(-y^2/4\nu t'\right)\frac{1}{2\sqrt{\nu t'}}dt'\right\}=\dot{u}\left\{-\frac{1}{\sqrt{\pi}}\int_0^t\frac{\exp\left(-y^2/4\nu t'\right)}{\sqrt{\nu t'}}dt'\right\},$$

$$\nu\frac{\partial^2 u}{\partial y^2}=\nu\frac{\dot{u}}{\sqrt{\pi}}\frac{\partial}{\partial y}\left\{-\int_0^t\frac{\exp\left(-y^2/4\nu t'\right)}{\sqrt{\nu t'}}dt'\right\}=\frac{\dot{u}y}{2\sqrt{\nu\pi}}\int_0^t\frac{\exp\left(-y^2/4\nu t'\right)}{t'^{3/2}}dt'.$$

Manipulate the final relationship for the second $y$-derivative to introduce the integration variable $\xi=y/2\sqrt{\nu t'}$ using $d\xi=\left(y/2\sqrt{\nu}\right)t'^{-3/2}(-1/2)dt'=-ydt/4\sqrt{\nu}t'^{3/2}$ to find:

$$\nu\frac{\partial^2 u}{\partial y^2} = -\frac{2\dot{u}}{\sqrt{\pi}}\int_0^t \exp\left(-y^2/4\nu t'\right)\left(\frac{-y\,dt'}{4\sqrt{\nu}\,t'^{3/2}}\right) = -\frac{2\dot{u}}{\sqrt{\pi}}\int_\infty^{y/2\sqrt{\nu t}} \exp\left(-\xi^2\right)d\xi$$

$$= +\frac{2\dot{u}}{\sqrt{\pi}}\int_{y/2\sqrt{\nu t}}^{\infty} \exp\left(-\xi^2\right)d\xi = +\frac{2\dot{u}}{\sqrt{\pi}}\int_0^{\infty} \exp\left(-\xi^2\right)d\xi - \frac{2\dot{u}}{\sqrt{\pi}}\int_0^{y/2\sqrt{\nu t}} \exp\left(-\xi^2\right)d\xi$$

$$= \dot{u}\left(1 - \frac{2}{\sqrt{\pi}}\int_0^{y/2\sqrt{\nu t}} \exp\left(-\xi^2\right)d\xi\right) = \dot{u}\left(1 - \text{erf}\left(y/2\sqrt{\nu t}\right)\right) = \frac{\partial u}{\partial t}$$

The final equality follows from the determination of $\partial u/\partial t$ above; thus, the velocity solution obtained by analogy does solve the field equation.

f) Start from the part e) solution and insert the definition of the error function to find:

$$u(y,t) = \dot{u}\int_0^t \left[1 - \frac{2}{\sqrt{\pi}}\int_0^{y/2\sqrt{\nu t'}} \exp(-\xi^2)d\xi\right]dt'.$$

Set $\eta' = y/\sqrt{\nu t'}$ so that $d\eta' = -\left(y/2\sqrt{\nu t'^3}\right)dt' = -\left(\nu/2y^2\right)\eta'^3 dt'$ and $dt' = -2\left(y^2/\nu\eta'^3\right)d\eta'$, and use this in the equation for $u(y, t)$:

$$u(y,t) = \dot{u}\int_\infty^\eta \left[1 - \frac{2}{\sqrt{\pi}}\int_0^{\eta'/2} \exp(-\xi^2)d\xi\right]\left(\frac{-2y^2}{\nu\eta'^3}\right)d\eta'.$$

Multiply by $t$ outside the integral, divide by $t$ inside the integral, and recognize the definition of $\eta$ to determine the form of the similarity function:

$$\frac{u(y,t)}{\dot{u}t} = f(\eta) = -2\eta^2\int_\infty^\eta \left[1 - \frac{2}{\sqrt{\pi}}\int_0^{\eta'/2} \exp(-\xi^2)d\xi\right]\frac{d\eta'}{\eta'^3} = -2\eta^2\int_\infty^\eta \left[1 - erf\left(\eta'/2\right)\right]\frac{d\eta'}{\eta'^3}.$$

The first and second derivatives of this function are needed to determine if it solves the part d) equation. These derivatives are:

$$\frac{df}{d\eta} = -4\eta\int_\infty^\eta \left[1 - erf\left(\frac{\eta'}{2}\right)\right]\frac{d\eta'}{\eta'^3} - 2\eta^2\left[1 - erf\left(\frac{\eta}{2}\right)\right]\frac{1}{\eta^3} = \frac{2}{\eta}\left(f - 1 + erf\left(\frac{\eta}{2}\right)\right), \text{ and}$$

$$\frac{d^2 f}{d\eta^2} = -\frac{2}{\eta^2}\left(f - 1 + erf\left(\frac{\eta}{2}\right)\right) + \frac{2}{\eta}\left(\frac{df}{d\eta} + \frac{2}{\sqrt{\pi}}e^{-\eta^2/4}\cdot\frac{1}{2}\right)$$

$$= -\frac{2}{\eta^2}\left(f - 1 + erf\left(\frac{\eta}{2}\right)\right) + \frac{2}{\eta}\left(\frac{2}{\eta}\left(f - 1 + erf\left(\frac{\eta}{2}\right)\right) + \frac{1}{\sqrt{\pi}}e^{-\eta^2/4}\right)$$

$$= \frac{2}{\eta^2}f - \frac{2}{\eta^2}\left[1 - erf\left(\frac{\eta}{2}\right)\right] + \frac{2}{\eta\sqrt{\pi}}e^{-\eta^2/4},$$

where the second equality for the second derivative follows from substituting in the first derivative result. Use these derivative results to assemble the part d) equation:

$$\frac{d^2 f}{d\eta^2} + \frac{\eta}{2}\frac{df}{d\eta} - f = \frac{2}{\eta^2}f - \frac{2}{\eta^2}\left[1 - erf\left(\frac{\eta}{2}\right)\right] + \frac{2}{\eta\sqrt{\pi}}e^{-\eta^2/4} + \left(f - 1 + erf\left(\frac{\eta}{2}\right)\right) - f$$

$$= \frac{2}{\eta^2}f - \left(\frac{2}{\eta^2} + 1\right)\left[1 - erf\left(\frac{\eta}{2}\right)\right] + \frac{2}{\eta\sqrt{\pi}}e^{-\eta^2/4}.$$

Consider the first term on the right and integrate by parts twice to find:

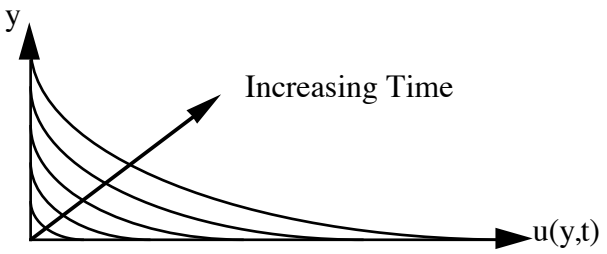
$$\frac{2}{\eta^2} f = -4\int_{\infty}^{\eta}\left[1-erf\left(\frac{\eta'}{2}\right)\right]\frac{d\eta'}{\eta'^3} = \frac{2}{\eta^2}\left[1-erf\left(\frac{\eta}{2}\right)\right] - \frac{2}{\sqrt{\pi}}\frac{e^{-\eta^2/4}}{\eta} - \frac{2}{\sqrt{\pi}}\left[\frac{1}{2}\int_{\infty}^{\eta} e^{-\eta'^2/4} d\eta'\right].$$

The final term on the right can be converted into $1 - erf(\eta/2)$ by changing the integration variable to $\xi = \eta'/2$ and breaking the range of integration, $\infty$ to $\eta$, into two intervals, $\infty$ to 0, and 0 to $\eta$. Thus, the reassembled part d) differential equation becomes:

$$\frac{d^2 f}{d\eta^2} + \frac{\eta}{2}\frac{df}{d\eta} - f = \frac{2}{\eta^2}\left[1-erf\left(\frac{\eta}{2}\right)\right] - \frac{2}{\sqrt{\pi}}\frac{e^{-\eta^2/4}}{\eta} + \left[1-erf\left(\frac{\eta}{2}\right)\right] - \left(\frac{2}{\eta^2}+1\right)\left[1-erf\left(\frac{\eta}{2}\right)\right] + \frac{2}{\eta\sqrt{\pi}} e^{-\eta^2/4}$$
$$= 0.$$

Here final equality occurs because there is term-by-term cancellation on the right side. Thus, the function $f(\eta)$ determined above is a valid solution of the part d) equation.

g) The velocity disturbance spreads linearly with time in flow direction, and in proportion to $\sqrt{t}$ in the vertical direction. The contours shown to the right are for quadratically spaced time increments.



**Exercise 9.32**. a) When $z$ is complex, the small-argument expansion of the zeroth-order Bessel function $J_o(z) = 1 - \frac{1}{4}z^2 + \ldots$ remains valid. Use this to show that (9.43) reduces to (9.6) as $\omega \to 0$ when $dp/dz = \Delta p/L$. The next term in the series is $\frac{1}{64}z^4$. At what value of $a/\sqrt{\nu/\omega}$ is the magnitude of this term equal to 5% of the second term.
b) When $z$ is complex, the large-argument expansion of the zeroth-order Bessel function $J_o(z) \cong (2/\pi z)^{1/2}\cos\left[z - \frac{1}{4}\pi\right]$ remains valid for $|\arg(z)| < \pi$. Use this to show that (9.43) reduces to the velocity profile of a viscous boundary layer on a plane wall beneath an oscillating flow as $\omega \to \infty$:

$$u_z(y,t) = -\frac{\Delta p}{\rho\omega L}\left[\sin(\omega t) - \exp\left\{-y\sqrt{\frac{\omega}{2\nu}}\right\}\sin\left(\omega t - y\sqrt{\frac{\omega}{2\nu}}\right)\right],$$

where $y$ is the distance from the tube wall, $R = a - y$, $y << a$, and $dp/dz = \Delta p/L$.

**Solution 9.32**. a) Start from (9.43):

$$u_z(R,t) = \mathrm{Re}\left\{i\frac{\Delta p}{\omega\rho L}\left[1 - J_o\left(\frac{i^{3/2}R}{\sqrt{\nu/\omega}}\right)\Big/ J_o\left(\frac{i^{3/2}a}{\sqrt{\nu/\omega}}\right)\right]e^{i\omega t}\right\}, \text{ and}$$

use the small argument form of $J_o$ for the limit $\omega \to 0$:

$$\lim_{\omega\to 0} u_z(R,t) = \lim_{\omega\to 0}\mathrm{Re}\left\{i\frac{\Delta p}{\omega\rho L}\left[1 - \frac{1 - \frac{1}{4}\left(\frac{i^{3/2}R}{\sqrt{\nu/\omega}}\right)^2 + \ldots}{1 - \frac{1}{4}\left(\frac{i^{3/2}a}{\sqrt{\nu/\omega}}\right)^2 + \ldots}\right]e^{i\omega t}\right\} = \lim_{\omega\to 0}\mathrm{Re}\left\{i\frac{\Delta p}{\omega\rho L}\left[1 - \frac{1 + \frac{i\omega R^2}{4\nu} + \ldots}{1 + \frac{i\omega a^2}{4\nu} + \ldots}\right]e^{i\omega t}\right\}.$$

Continue simplifying:

$$\lim_{\omega\to 0} u_z(R,t) = \lim_{\omega\to 0}\mathrm{Re}\left\{i\frac{\Delta p}{\omega\rho L}\left[1 - \left(1 + \frac{i\omega R^2}{4\nu} - \frac{i\omega a^2}{4\nu} + \ldots\right)\right]e^{i\omega t}\right\}$$

$$= \lim_{\omega\to 0}\mathrm{Re}\left\{i\frac{\Delta p}{\omega\rho L}\left[-\frac{i\omega R^2}{4\nu} + \frac{i\omega a^2}{4\nu}\right]e^{i\omega t}\right\}$$

$$= \mathrm{Re}\left\{\frac{\Delta p}{\rho L}\left[+\frac{R^2}{4\nu} - \frac{a^2}{4\nu}\right]\right\} = \frac{1}{4\mu}\left(-\frac{\Delta p}{L}\right)\left(a^2 - R^2\right)$$

and this is the same as (9.6) when the pressure gradient is $\Delta p/L$.

To determine when $\frac{1}{64}z^4$ is 5% of $\frac{1}{4}z^2$, set $(0.05)\frac{1}{4}z^2 = \frac{1}{64}z^4$ and determine $z$. The solution is $|z| = a/\sqrt{\nu/\omega} = \sqrt{0.05(64)/4} = 0.894$.

b) Here, $z = \frac{i^{3/2}R}{\sqrt{\nu/\omega}} = \left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\frac{R}{\sqrt{\nu/\omega}}$, so

$$\cos\left(z-\frac{\pi}{4}\right)=\frac{1}{2}\left[\exp\left\{i\left(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\frac{R}{\sqrt{\nu/\omega}}-i\frac{\pi}{4}\right\}+\exp\left\{-i\left(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\frac{R}{\sqrt{\nu/\omega}}+i\frac{\pi}{4}\right\}\right]$$

$$=\frac{1}{2}\left[\exp\left\{\left(-\frac{i}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)\frac{R}{\sqrt{\nu/\omega}}-i\frac{\pi}{4}\right\}+\exp\left\{\left(\frac{i}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)\frac{R}{\sqrt{\nu/\omega}}+i\frac{\pi}{4}\right\}\right]$$

When $\omega\to\infty$, the first term becomes exponentially small, so

$$\cos\left(z-\frac{\pi}{4}\right)\cong\frac{1}{2}\exp\left\{\left(\frac{i}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)\frac{R}{\sqrt{\nu/\omega}}+i\frac{\pi}{4}\right\}\quad\text{as}\quad\omega\to\infty.$$

Now use $R = a - y$ in the above expression and collect like factors:

$$\cos\left(z-\frac{\pi}{4}\right)\cong\frac{1}{2}\exp\left\{i\left(\frac{a-y}{\sqrt{2\nu/\omega}}+\frac{\pi}{4}\right)+\frac{a-y}{\sqrt{2\nu/\omega}}\right\}\quad\text{as}\quad\omega\to\infty.$$

or:

$$\cos\left(z-\frac{\pi}{4}\right)\cong\frac{e^{i\pi/4}}{2}\exp\left(\frac{a(1+i)}{\sqrt{2\nu/\omega}}\right)\exp\left(\frac{-(1+i)}{\sqrt{2\nu/\omega}}y\right)\quad\text{as}\quad\omega\to\infty.$$

So, in this limit:

$$\frac{J_o\left(\frac{i^{3/2}R}{\sqrt{\nu/\omega}}\right)}{J_o\left(\frac{i^{3/2}a}{\sqrt{\nu/\omega}}\right)}=\frac{\sqrt{\frac{2}{\pi}\frac{\sqrt{\nu/\omega}}{i^{3/2}(a-y)}}\frac{e^{i\pi/4}}{2}\exp\left(\frac{a(1+i)}{\sqrt{2\nu/\omega}}\right)\exp\left(\frac{-(1+i)}{\sqrt{2\nu/\omega}}y\right)}{\sqrt{\frac{2}{\pi}\frac{\sqrt{\nu/\omega}}{i^{3/2}a}}\frac{e^{i\pi/4}}{2}\exp\left(\frac{a(1+i)}{\sqrt{2\nu/\omega}}\right)}$$

$$=\sqrt{\frac{a}{a-y}}\exp\left(\frac{-(1+i)}{\sqrt{2\nu/\omega}}y\right)$$

$$\cong\exp\left(\frac{-(1+i)}{\sqrt{2\nu/\omega}}y\right)$$

where the final approximate equality holds when $y << a$. Now substitute this approximate ratio of Bessel functions into (9.43) to find:

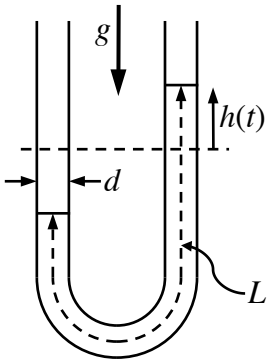
$$u_z(y,t)=\mathrm{Re}\left\{\frac{i\Delta p}{\omega\rho L}\left[1-\exp\left(\frac{-(1+i)y}{\sqrt{2\nu/\omega}}\right)\right]e^{i\omega t}\right\}=\mathrm{Re}\left\{\frac{i\Delta p}{\omega\rho L}\left[e^{i\omega t}-\exp\left(\frac{-y}{\sqrt{2\nu/\omega}}\right)\exp\left(i\omega t-\frac{iy}{\sqrt{2\nu/\omega}}\right)\right]\right\}.$$

Take the real part to reach:

$$u_z(y,t)=-\frac{\Delta p}{\omega\rho L}\left[\sin(\omega t)-\exp\left(\frac{-y}{\sqrt{2\nu/\omega}}\right)\sin\left(\omega t-\frac{y}{\sqrt{2\nu/\omega}}\right)\right],$$

and this is the desired result.

**Exercise 9.33**. A round tube bent into a U-shape having inner diameter $d$ holds a column of liquid with overall length $L$. Initially the column of liquid is pushed upward on the right side of the U-tube and downward on the left side of the U-tube so that the two liquid surfaces are a vertical distance $2h_o$ apart. If the liquid has density $\rho$ and viscosity $\mu$, and the column is released from rest, find and solve an approximate ordinary differential equation that describes the subsequent damped oscillations of $h(t)$, the liquid height above equilibrium in the right side of the U-tube, assuming that the flow profile at any time throughout the tube is parabolic. Under what condition(s) is this approximate solution valid? Will oscillations occur in this parameter regime?




**Solution 9.33**. This is the fluid mechanical pendulum with viscous effects included. Use Newton's second law for the mass of fluid in the water column. The unbalanced gravitational force tends to decrease $h(t)$ and is $-2\rho g h(\pi d^2/4)$ and the wall shear stress opposes the motion. Thus:

$$\rho\frac{\pi}{4}d^2L\frac{d^2h}{dt^2} = -\pi dL\tau_w - 2\rho g\frac{\pi}{4}d^2h\,,$$

where $\tau_w$ is presumed to be negative. Here $dh/dt$ is the cross-section-average fluid velocity, so from the results for steady flow in a round tube:

$$V_{ave} = \frac{dh}{dt} = -\frac{d^2}{32\mu}\frac{dp}{dz} = -\frac{d^2}{32\mu}\left(\frac{4\tau_w}{d}\right),$$

where the final equality follows from (9.8). Use this results to eliminate $\tau_w$ from the first equation to reach:

$$\rho\frac{\pi}{4}d^2L\frac{d^2h}{dt^2} = -\pi dL\left(\frac{8\mu}{d}\frac{dh}{dt}\right) - 2\rho g h\frac{\pi}{4}d^2h\,.$$

Divide by the coefficient of $d^2h/dt^2$, and rearrange the terms:

$$\frac{d^2h}{dt^2} + \frac{32\mu}{\rho d^2}\frac{dh}{dt} + \frac{2g}{L}h = 0\ . \qquad (\$)$$

This equation can be solved by assuming an exponential solution: $h(t) = h_o e^{-mt}$, which leads to an algebraic equation for $m$:

$$m^2 - \left(\frac{32\nu}{d^2}\right)m + \frac{2g}{L} = 0 \quad \text{or} \quad m = \frac{16\nu}{d^2}\left(-1 \pm \sqrt{1 - \frac{gd^4}{128\nu^2L}}\right),$$

where $\nu = \mu/\rho$. Thus, the general solution is:

$$h(t) = A_+ \exp\left\{\frac{16\nu t}{d^2}\left(-1+\sqrt{1-\frac{gd^4}{128\nu^2 L}}\right)\right\} + A_- \exp\left\{\frac{16\nu t}{d^2}\left(-1-\sqrt{1-\frac{gd^4}{128\nu^2 L}}\right)\right\} .$$

The constants can be determined from the initial conditions, $h(0) = h_o$, and $dh/dt = 0$, which lead to two algebraic equations:

$$h_o = A_+ + A_- \text{ , and } 0 = A_+ \frac{16\nu}{d^2}\left(-1+\sqrt{1-\frac{gd^4}{128\nu^2 L}}\right) + A_- \frac{16\nu}{d^2}\left(-1-\sqrt{1-\frac{gd^4}{128\nu^2 L}}\right) .$$

Thus, the constants are:

$$A_+ = \frac{h_o}{2}\left(1+\frac{1}{\sqrt{1-\psi}}\right), \text{ and } A_- = \frac{h_o}{2}\left(1-\frac{1}{\sqrt{1-\psi}}\right),$$

where $\psi = gd^4/128\nu^2 L$. The final solution is:

$$h(t) = \frac{h_o}{2}\left(1+\frac{1}{\sqrt{1-\psi}}\right)\exp\left\{\frac{16\nu t}{d^2}\left(-1+\sqrt{1-\psi}\right)\right\} + \frac{h_o}{2}\left(1-\frac{1}{\sqrt{1-\psi}}\right)\exp\left\{\frac{16\nu t}{d^2}\left(-1-\sqrt{1-\psi}\right)\right\} . \quad (\&)$$

This approximate solution will be valid when $d << L$ so the nearly the entire water column is not influenced by either free surface, and when the flow's oscillations are slow enough so that the full-developed parabolic flow profile is valid. Stated simply, this second requirement is:

$$\frac{d^2}{\nu T} << 1 ,$$

where $d$ represents the requisite viscous diffusion distance and $T$ is the time scale of the flow. If the flow oscillates, the period of this oscillation will scale with $[L/g]^{1/2}$. Thus, the flow development time requirement can be stated as:

$$\frac{d^2}{\nu}\sqrt{\frac{g}{L}} << 1 .$$

Thus, $\psi << 1$, so the solution provided above suggests that the flow will <u>not</u> oscillate and will instead be over-damped. In this case, the above solution simplifies to:

$$h(t) \cong h_o \exp\left\{-\frac{gd^2}{16\nu L}t\right\} ,$$

This solution can be obtained directly from ($) by dropping the acceleration term and solving.

**Exercise 9.34**. Suppose a line vortex of circulation Γ is suddenly introduced into a fluid at rest at $t = 0$. Show that the solution is $u_\theta(r,t) = \left(\Gamma/2\pi r\right)\exp\left\{-r^2/4\nu t\right\}$. Sketch the velocity distribution at different times. Calculate and plot the vorticity, and observe how it diffuses outward.

**Solution 9.34**. The solution to this problem is very similar to the decay of a line vortex (see Example 9.8). In two-dimensional $(r,\theta)$-polar coordinates, the governing equation is:

$$\frac{\partial u_\theta}{\partial t} = \nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_\theta\right)\right)\right].$$

The boundary conditions on the velocity $u_\theta(r, t)$ are

$$u_\theta(r,0^+) = 0,\ u_\theta(r,\infty) = \Gamma/2\pi r, \text{ and } u_\theta(\infty, t) = 0.$$

In this case the second boundary condition suggests a similarity solution of the form:

$$u_\theta = \frac{\Gamma}{2\pi r} f\left(\eta\right) = \frac{\Gamma}{2\pi r} f\left(\frac{r}{\sqrt{\nu t}}\right).$$

For this solution form the time and radial derivatives are:

$$\frac{\partial u_\theta}{\partial t} = \frac{\Gamma}{2\pi r}\frac{df}{d\eta}\frac{\partial \eta}{\partial t} = \frac{\Gamma}{2\pi r}\frac{df}{d\eta}\left(-\frac{\eta}{2t}\right) = -\frac{\Gamma}{2\pi r}\left(\frac{\eta}{2t}\right)\frac{df}{d\eta},\ \frac{\partial(ru_\theta)}{\partial r} = \frac{\Gamma}{2\pi}\frac{df}{d\eta}\frac{\partial \eta}{\partial r} = \frac{\Gamma}{2\pi}\frac{df}{d\eta}\left(\frac{1}{\sqrt{\nu t}}\right), \text{ and}$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_\theta\right)\right) = \frac{\partial}{\partial r}\left(\frac{\Gamma}{2\pi r}\frac{1}{\sqrt{\nu t}}\frac{df}{d\eta}\right) = -\frac{\Gamma}{2\pi r^2}\frac{1}{\sqrt{\nu t}}\frac{df}{d\eta} + \frac{\Gamma}{2\pi r}\frac{1}{\nu t}\frac{d^2 f}{d\eta^2}.$$

Reassemble the governing equation and divide out the common factor of $\Gamma/2\pi r$:

$$-\left(\frac{\eta}{2t}\right)\frac{df}{d\eta} = -\frac{\nu}{r\sqrt{\nu t}}\frac{df}{d\eta} + \frac{\nu}{\nu t}\frac{d^2 f}{d\eta^2} = -\frac{1}{t}\left(\frac{1}{\eta}\frac{df}{d\eta} - \frac{d^2 f}{d\eta^2}\right).$$

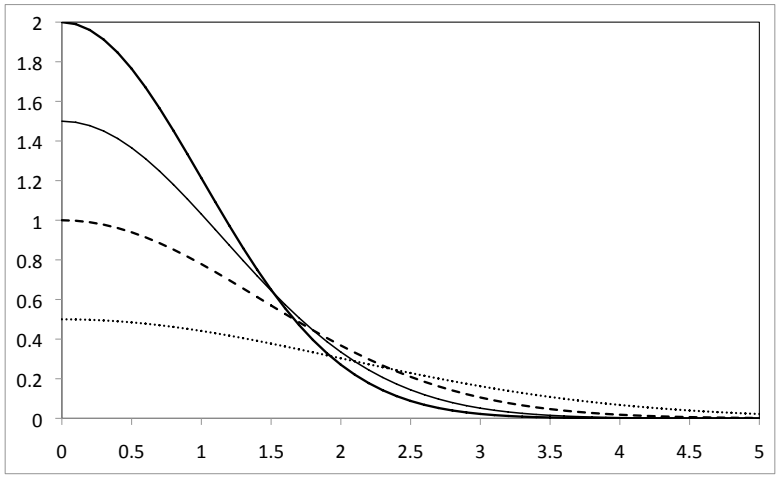
Multiply by $t$ and put the second derivative on the left: $\dfrac{d^2 f}{d\eta^2} = \left(-\dfrac{\eta}{2} + \dfrac{1}{\eta}\right)\dfrac{df}{d\eta}$.

Integrate to find: $\ln\dfrac{df}{d\eta} = -\dfrac{\eta^2}{4} + \ln\eta + const.$ Exponentiate $\dfrac{df}{d\eta} = e^{const}\cdot\eta\exp\left\{-\eta^2/4\right\}$ and integrate again:

$$f = A + B\exp\left\{-\eta^2/4\right\}.$$

The constants $A$ and $B$ can be determined from the boundary conditions: $f(0) = 1$, and $f(\infty) = 1$; $A = 0$, and $B = 1$. Thus, the velocity field is:

$$u_\theta(r,t) = \frac{\Gamma}{2\pi r}\exp\left\{-\frac{r^2}{4\nu t}\right\}.$$




In this flow the $z$-component of the vorticity is the only non-zero component.

$$\omega_z(r,t) = \frac{1}{r}\frac{\partial}{\partial r}\left(ru_\theta\right) - \frac{1}{r}\frac{\partial u_r}{\partial \theta} = \frac{\Gamma}{4\pi\nu t}\exp\left\{-\frac{r^2}{4\nu t}\right\} = \frac{\Gamma}{4\pi\nu t}\exp\left\{-\frac{\eta^2}{4}\right\}.$$

The plot above shows $\omega_z$ (vertical axis) vs. $r$ (horizontal axis) at four different times. With increasing time, the vorticity ar $r = 0$ decreases but it spreads outward in the radial direction.

**Exercise 9.35**. Following Taylor and Green (1937), consider the two-dimensional vortex flow field with constant density $\rho$:

$$\mathbf{u} = (u,v) = \big(A\sin(kx)\cos(ky), B\cos(kx)\sin(ky)\big) .$$

a) If the flow is steady and inviscid, and $A$ and $B$ are constants, explicitly determine the pressure, $p(x,y)$, in terms of $x$, $y$, $A$, $\rho$, and $k$ from (4.10) and (9.1) in two dimensions.
b) If the flow field is unsteady and viscous (with viscosity $\mu$), $A$ and $B$ are functions of time $t$, and $A = A_o$ at $t = 0$, determine $A(t)$, $B(t)$, and $p(x,y,t)$ so that the given $\mathbf{u}$ is an exact solution of (4.10) and (9.1) in two dimensions.
c) How long does it take for $A(t)$ to fall to $A_o/2$? Does the parametric dependence of this decay time follow typical diffusion scaling?

**Solution 9.35**. a) Here $\rho$ = const. so (4.10) and (9.1) simplify to:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 ,$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),$$

$$\text{and} \quad \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right).$$

First use the continuity equation to find: $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = Ak\cos\big(kx\big)\cos\big(ky\big) + Bk\cos\big(kx\big)\cos\big(ky\big) = 0$ ,

which implies $B = -A$. Without the unsteady & viscous terms, the $x$- and $y$-momentum equations are: $u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}$ , and $u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y}$ . For the given velocity field these imply:

$$A^2 k\sin\big(kx\big)\cos(kx)\cos^2\big(ky\big) - BA\cos\big(kx\big)\sin\big(kx\big)\sin^2\big(ky\big) = -\big(1/\rho\big)\big(\partial p/\partial x\big) , \text{ and}$$

$$-ABk\sin^2\big(kx\big)\cos\big(ky\big)\sin(ky) + B^2\cos^2\big(kx\big)\sin(ky)\cos(ky) = -\big(1/\rho\big)\big(\partial p/\partial y\big) .$$

These equations can be simplified using $\sin^2(\,) + \cos^2(\,) = 1$, and the relationship $B = -A$ to find:

$$A^2 k\big[\sin\big(kx\big)\cos(kx)\big] = \frac{A^2 k}{2}\sin(2kx) = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad \text{and} \quad A^2 k\big[\cos\big(ky\big)\sin(ky)\big] = \frac{A^2 k}{2}\sin(2ky) = -\frac{1}{\rho}\frac{\partial p}{\partial y} .$$

Integrating both final equalities leads to:

$$p = \rho\frac{A^2}{4}\cos(2kx) + f(y) \quad \text{and} \quad p = \rho\frac{A^2}{4}\cos(2ky) + g(x) ,$$

where $f$ and $g$ are single-variable *functions of integration*. Thus, the pressure field is:

$$p(x,y) = \rho\frac{A^2}{4}\big[\cos(2kx) + \cos(2ky)\big] + p_o ,$$

where $p_o$ is an undetermined constant.
b) When the flow field is unsteady and viscous, the continuity equation result from part b), $B = -A$, still holds. Plus, the analysis leading to the pressure field of part b) remains the same with the function $A(t)$ replacing the constant $A$. So, for the given velocity field, the remnant of the $x$-momentum equation is:

$$\frac{\partial u}{\partial t} = \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \text{ or } \left(\frac{dA}{dt}\right)\sin(kx)\cos(ky) = -\nu A k^2\left(\sin(kx)\cos(ky) + \sin(kx)\cos(ky)\right),$$

where $\nu = \mu/\rho$. Dividing out the common trig-function product leads to $\frac{dA}{dt} = -2\nu A k^2$, which is readily solved to find:

$$A(t) = -B(t) = A(0)\exp\left\{-2\nu k^2 t\right\}.$$

In this case the pressure field is:

$$p(x,y,t) = \rho\frac{A^2(0)}{4}\exp\left\{-4\nu k^2 t\right\}\left[\cos(2kx) + \cos(2ky)\right] + p_o,$$

c) To determine the half-life ($t_{1/2}$) of the flow, set:

$$1/2 = \exp\{-k^2\nu\, t_{1/2}\}, \text{ to reach: } t_{1/2} = \ln(2)/k^2\nu.$$

Thus, the decay time of this flow does follow typical diffusion scaling. This result also indicates that Taylor-Green flow with a high wave number decays more quickly than an equivalent flow with a low wave number.

**Exercise 9.36**. Obtain several liquids of differing viscosity (water, cooking oil, pancake syrup, shampoo, etc.). Using an eyedropper, a small spoon, or your finger, place a drop of each on a smooth vertical surface (a bathroom mirror perhaps) and measure how far the drops have moved or extended in a known period of time (perhaps a minute or two). Try to make the mass of all the drops equal. Using dimensional analysis, determine how the drop-sliding distance depends on the other parameters. Does this match your experimental results?

**Solution 9.36**. The experiments yield the following three qualitative results. 1) The drops of the more viscous liquids do not slide as quickly as those of the less viscous liquids. 2) The drops slow down as time increases. 3) Bigger drops go farther. Water does not behave like oils or liquid soaps because its surface tension is large enough to influence the motion of the sliding drop. Here, we wish to use the simplest possible dimensional analysis so surface tension will be ignored.

• Boundary condition & material parameters: $\mu$ = viscosity of the fluid
$W$ = weight of the drop
$t$ = sliding time
Solution parameter: $s$ = sliding distance

• Create the parameter matrix:

| | $s$ | $t$ | $W$ | $\mu$ |
|---|---|---|---|---|
| Mass: | 0 | 0 | 1 | 1 |
| Length: | 1 | 0 | 1 | -1 |
| Time: | 0 | 1 | -2 | -1 |

This matrix has rank three.

• Determine the number of dimensionless groups: 4 parameters - 3 dimensions = 1 group

• Construct the dimensionless group: $\Pi_1 = Wt/\mu s^2$

• Write a dimensionless law: $s = const\sqrt{Wt/\mu}$. This simple law accounts for all three "bathroom mirror" observations!

**Exercise 9.37**. A drop of an incompressible viscous liquid is allowed to spread on a flat horizontal surface under the action of gravity. Assume the drop spreads in an axisymmetric fashion and use cylindrical coordinates ($R$, $\varphi$, $z$). Ignore the effects of surface tension.

a) Show that conservation of mass implies: $\frac{\partial h}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\left(R\int_o^h u dz\right) = 0$, where $u = u(R,z,t)$ is the horizontal velocity within the drop, and $h = h(R,t)$ is the thickness of the spreading drop.

b) Assume that the lubrication approximation applies to the horizontal velocity profile, i.e. $u(R,z,t) = a(R,t) + b(R,t)z + c(R,t)z^2$, apply the appropriate boundary conditions on the upper and lower drop surfaces, and require a pressure & shear-stress force balance within a differential control volume $h(R,t)RdRd\theta$ to show that: $u(R,z,t) = -\frac{g}{2\nu}\frac{\partial h}{\partial R}z(2h - z)$.

c) Combine the results of a) and b) to find $\frac{\partial h}{\partial t} = \frac{g}{3\nu R}\frac{\partial}{\partial R}\left(Rh^3\frac{\partial h}{\partial R}\right)$.

d) Assume a similarity solution: $h(R,t) = \frac{A}{t^n}f(\eta)$ with $\eta = \frac{BR}{t^m}$, use the result of part c) and $2\pi\int_o^{R_{\max}(t)} h(R,t)RdR = V$ where $R_{\max}(t)$ is the radius of the spreading drop and $V$ is the initial volume of the drop to determine $m$ = 1/8, $n$ = 1/4, and a single nonlinear ordinary differential equation for $f(\eta)$ involving only $A$, $B$, $g/\nu$, and $\eta$. You need not solve this equation for $f$. [Given that $f \to 0$ as $\eta \to \infty$, there will be a finite value of $\eta$ for which $f$ is effectively zero. If this value of $\eta$ is $\eta_{\max}$ then the radius of the spreading drop, $R(t)$, will be: $R_{\max}(t) = \eta_{\max}t^m/B$.]

**Solution 9.37**. a) Consider a stationary "ring" control volume that has inner radius of $R$, outer radius of $R + dR$, and height $h(R,t)$. There is no flow out through the underside of the CV, but all three other sides contribute a term to the mass balance.

$$-2\pi R\int_0^{h(R,t)} u dz + 2\pi(R + dR)\int_0^{h(R+dR,t)} u dz + 2\pi(R + dR/2)\frac{\partial h}{\partial t}dR = 0$$

(inner surface) (outer surface) (top surface)

Divide by $2\pi$, drop the term containing a $(dR)^2$ and divide by $dR$ to find:

$$R\frac{\partial h}{\partial t} + \frac{1}{dR}\left((R + dR)\int_0^{h(R+dR,t)} u dz - R\int_0^{h(R,t)} u dz\right) = 0$$

Take the limit as $dR \to 0$, recognize the definition of a derivative, and divide by $R$ to obtain:

$$\frac{\partial h}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\left(R\int_o^h u dz\right) = 0.$$

b) Start with $u(R,z,t) = a(R,t) + b(R,t)z + c(R,t)z^2$ and apply the no-slip, wall-shear, and free-surface boundary conditions:

i) no slip: $u(R,0,t) = 0$ –> $a(R,t) = 0$,

ii) wall shear: $\mu\left(\frac{\partial u}{\partial z}\right)_{z=0} = \tau_w$ –> $\mu b(R,t) = \tau_w$, and

iii) free surface: $\mu\left(\frac{\partial u}{\partial z}\right)_{z=h} = 0$ –> $b(R,t) + 2c(R,t)h = 0$,

Therefore: $u(R,z,t) = \frac{\tau_w}{2h\mu}z(2h - z)$. Now consider a creeping-flow force balance in the radial direction for the differential control volume $h(R,t)RdRd\theta$.

$$\int_0^{h(R,t)} p(R)Rd\theta dz - \int_0^{h(R+dR,t)} p(R+dR)(R+dR)d\theta dz - \tau_w RdRd\theta = 0$$

Here take atmospheric pressure as the zero or reference pressure level. The pressure that matters will be hydrostatic, $p(R,t) = \rho g\big(h(R,t) - z\big)$ so that the above equation becomes:

$$\frac{\rho g h^2(R,t)}{2} Rd\theta - \frac{\rho g h^2(R+dR,t)}{2}(R+dR)d\theta - \tau_w RdRd\theta = 0.$$

Divide this by $Rd\theta$, drop the higher order term, and rearrange to form the definition of a derivative:

$$-\frac{1}{dR}\left(\frac{\rho g h^2(R+dR,t)}{2} - \frac{\rho g h^2(R,t)}{2}\right) = \tau_w$$

Take the limit as $dR \to 0$ and perform the radial derivative:

$$\tau_w = -\frac{\partial}{\partial R}\left(\frac{\rho g h^2}{2}\right) = -\rho g h \frac{\partial h}{\partial R}.$$

Place $\nu = \mu/\rho$ and this wall-shear-stress-static-pressure-gradient relationship into the velocity profile formula to simplify it and eliminate $\tau_w$:

$$u(R,z,t) = -\frac{g}{2\nu}\frac{\partial h}{\partial R} z(2h - z).$$

c) Use the result of part b) and perform the integration specified in the result of part a)

$$\int_o^h u dz = -\frac{g}{2\nu}\frac{\partial h}{\partial R}\int_o^h z(2h-z)dz = -\frac{g}{2\nu}\frac{\partial h}{\partial R}\left(2h\frac{z^2}{2} - \frac{z^3}{3}\right)_0^h = -\frac{gh^3}{3\nu}\frac{\partial h}{\partial R}$$

Put this into the result of part a) and bring the constants outside the radial differentiation to find:

$$\frac{\partial h}{\partial t} - \frac{g}{3\nu R}\frac{\partial}{\partial R}\left(Rh^3\frac{\partial h}{\partial R}\right) = 0.$$

d) First consider a CV that entirely encloses the drop and moves with it. Thus, conservation of mass in this case implies: $\frac{d}{dt}\left(2\pi\int_0^{R_{\max}(t)} h(R,t)RdR\right) = 0$ because there are no fluxes crossing the CV boundary. Integrate once in time to find: $2\pi\int_0^{R_{\max}(t)} h(R,t)RdR = const$. The constant can be evaluated as $V$, the volume of the drop, at the initial time (or any other time); thus,

$$2\pi\int_0^{R_{\max}(t)} h(R,t)RdR = V.$$

Now use the proposed similarity solution form, $h(R,t) = \frac{A}{t^n} f(\eta)$ with $\eta = \frac{BR}{t^m}$, and perform the various differentiations:

$$\frac{\partial}{\partial t}h(R,t) = -n\frac{A}{t^{n+1}}f(\eta) + \frac{A}{t^n}f'(\eta)\left(\frac{-mBR}{t^{m+1}}\right) = -n\frac{A}{t^{n+1}}f(\eta) - m\frac{A}{t^{n+1}}\eta f'(\eta),$$

$$\frac{\partial}{\partial R}h(R,t) = \frac{A}{t^n}f'(\eta)\frac{B}{t^m} = \frac{AB}{t^{n+m}}f'(\eta),\ Rh^3\frac{\partial}{\partial r}h(R,t) = R\frac{A^4B}{t^{4n+m}}f^3(\eta)f'(\eta), \text{ so}$$

$$\frac{1}{R}\frac{\partial}{\partial R}\left(Rh^3\frac{\partial}{\partial R}h(R,t)\right) = \frac{A^4B}{Rt^{4n+m}}f^3(\eta)f'(\eta) + \frac{A^4B^2}{t^{4n+2m}}\left(3f^2(\eta)f'(\eta) + f^3(\eta)f''(\eta)\right)$$

$$= \frac{A^4B^2}{t^{4n+2m}\eta}f^3(\eta)f'(\eta) + \frac{A^4B^2}{t^{4n+2m}}\left(3f^2(\eta)f'(\eta) + f^3(\eta)f''(\eta)\right).$$

Thus, the field equation determined for part c) implies:

$$-n\frac{A}{t^{n+1}}f(\eta)-m\frac{A}{t^{n+1}}\eta f'(\eta)=\frac{g}{3\nu}\frac{A^4B^2}{t^{4n+2m}\eta}f^3(\eta)f'(\eta)+\frac{g}{3\nu}\frac{A^4B^2}{t^{4n+2m}}\left(3f^2(\eta)f'^2(\eta)+f^3(\eta)f''(\eta)\right),$$

which will allow a valid similarity solution when: $n+1=4n+2m$. The second relationship for *n* and *m* can be obtained from the conservation of volume requirement:

$$\int_0^{R_{\max}(t)}h(R,t)RdR=\frac{V}{2\pi}=\int_0^{R_{\max}(t)}\frac{A}{t^n}f(\eta)RdR=\frac{At^{2m}}{t^nB^2}\int_0^{\eta_{\max}}f(\eta)\eta d\eta, \qquad (*)$$

and so this second relationship is $2m=n$. Together these two relationships for *n* and *m* imply: $n=1/4$ and $m=1/8$, so that the remnant of the field equation is:

$$-\frac{1}{4}f(\eta)-\frac{1}{8}\eta f'(\eta)=\frac{gA^3B^2}{3\nu\eta}f^3(\eta)f'(\eta)+\frac{gA^3B^2}{3\nu}\left(3f^2(\eta)f'^2(\eta)+f^3(\eta)f''(\eta)\right) \qquad (**).$$

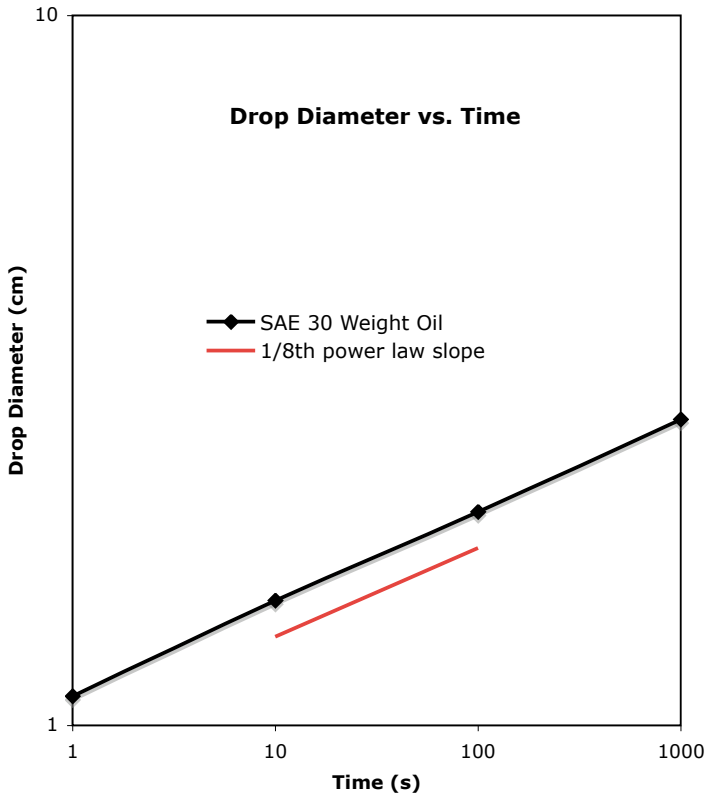
The constants *A* and *B* can be determined by simultaneous solution of (*) and (**) using $h(R_{\max}(t),t)=0$, and $\left(\partial h/\partial R\right)_{R=0}=0$ as boundary conditions for (**). However, this effort is beyond the scope of this exercise.

**Exercise 9.38**. Obtain a clean flat glass plate, a watch, a ruler, and some non-volatile oil that is more viscous than water. The plate and oil should be at room temperature. Dip the tip of one of your fingers in the oil and smear it over the center of the plate so that a thin bubble-free oil film covers a circular area ~10 to 15 cm in diameter. Set the plate on a horizontal surface and place a single drop of oil at the center of the oil-film area and observe how the drop spreads. Measure the spreading drop's diameter 1, 10, $10^2$, $10^3$, and $10^4$ seconds after the drop is placed on the plate. Plot your results and determine if the spreading drop diameter grows as $t^{1/8}$ (the predicted drop-diameter time dependence from the prior exercise) to within experimental error.

**Solution 9.38**. For this kitchen-or-garage experiment, the third author of this textbook used the glass plate from the front of a document frame and SAE 30-Weight motor oil to obtain the following measurements.

| time (s) | Drop Diameter (cm) |
|---|---|
| 1 | $1.1 \pm 0.2$ |
| 10 | $1.5 \pm 0.1$ |
| $10^2$ | $2.0 \pm 0.1$ |
| $10^3$ | $2.7 \pm 0.1$ |
| $10^4$ | — |

Here most of the uncertainty comes from the fact that the spreading droplet was not precisely circular. The first measurement at $t = 1$ second was more uncertain than the rest because starting the experiment and reading the ruler had be done nearly simultaneously so catching the diameter at just the right instant in time wasn't easy. The final measurement at $10^4$ seconds wasn't made because the drop had spread so far that its edge was not distinct. However, for the experimental results tabulated above and plotted to the right, there were no substantial differences from the theory. Any differences that may have been found between the theory and the experiments are likely caused by physical processes or disturbances that were not included in the theory. Evaporation, surface tension, contact angle phenomena, foreign matter on the surface, and surface roughness are all possible reasons that drop spreading measurements would be different from the theoretical predictions.

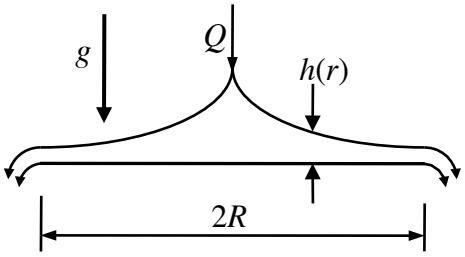


**Exercise 9.39**. A fine stream of a viscous fluid with density $\rho$ and viscosity $\mu$ falls slowly at a constant volume flow rate $Q$ onto the center of a flat horizontal circular disk of radius $R$. The fluid flows steadily under the action of gravity $g$ from the center of the disk to its edge in a layer of thickness $h(r)$, where $r$ is the radial coordinate. For the following items, assume $Q$ is constant, and apply the approximate boundary condition $h(R^+) = 0$, where $R^+$ is a radial location just beyond the edge of the disk.

a) Determine a scaling law for $h$ from dimensional analysis.

b) Using the lubrication approximation determine a formula for $h(r)$ that is valid for $0 < r < R$.

c) Increasing which parameters increases the thickness of the fluid layer on the disk.




**Solution 9.39**. a) There are six parameters ($h$, $r$, $Q$, $g$, $R$, $\rho$, and $\mu$) and all three fundamental dimensions are present so there will be 7 – 3 = 4 dimensionless parameters. Here $h$ is the solution parameter so put it in the first group, $\Pi_1 = h/R$. The second group is also a simple length scale ratio $\Pi_2 = r/R$. The other two groups are a Froude number $\Pi_3 = Q/[gR^5]^{1/2}$, and a Reynolds number $\Pi_4 = \rho Q/R\mu$. Thus: $h/R = f\left(r/R, Q/\sqrt{gR^5}, \rho Q/R\mu\right)$ is the scaling law.

b) The local velocity profile $u(r,z)$ is assumed parabolic in $z$ when a flow is analyzed with the lubrication approximation. In this case, the flow is driven by a static pressure gradient and the shear stress is zero on the free surface on $y = h$. Thus, from Example 9.5, the steady horizontal velocity profile will be: $u(r,z) = -\dfrac{g}{2\nu}\dfrac{\partial h}{\partial r} z(2h - z)$. The volume flux across the plate surface must be $Q$, so

$$Q = 2\pi r \int_{z=0}^{z=h} u(r,z)dz = -\frac{\pi g r}{\nu}\frac{\partial h}{\partial r}\int_{z=0}^{z=h} z(2h-z)dz = -\frac{\pi g r}{\nu}\frac{\partial h}{\partial r}\left(2h\frac{h^2}{2} - \frac{h^3}{3}\right) = -\frac{2\pi g r}{3\nu} h^3 \frac{\partial h}{\partial r}.$$

Treating the two ends of this extended equality as a differential equation leads to:

$$\int (1/r)dr = -(2\pi g/3\nu Q)\int h^3 dh \quad \text{or} \quad \ln r + C = -(2\pi g/3\nu Q)\left(h^4/4\right),$$

where $C$ is a constant. Applying $h(R^+) = 0$ produces $C = -\ln(R^+)$, and $h = \left[(6\nu Q/\pi g)\ln\left(R^+/r\right)\right]^{1/4}$.

c) The fluid layer will thicken when $\mu$, $Q$, or $R$ increase.

**Exercise 9.40**. An infinite flat plate located at $y = 0$ is stationary until $t = 0$ when it begins moving horizontally in the positive $x$-direction at a constant speed $U$. This motion continues until $t = T$ when the plate suddenly stops moving.
a) Determine the fluid velocity field, $u(y,t)$ for $t > T$. At what height above the plate does the peak velocity occur for $t > T$? [*Hint*: the governing equation is linear so superposition of solutions is possible.]
b) Determine the mechanical impulse $I$ (per unit depth and length) imparted to the fluid while the plate is moving: $I = \int_0^T \tau_w \, dt$.
c) As $t \to \infty$, the fluid slows down and eventually stops moving. How and where was the mechanical impulse dissipated? What is $t/T$ when 99% of the initial impulse has been lost?

**Solution 9.40**. a) For $0 < t < T$, the plate moves to the right and $u(y,t) = U\left(1 - \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\xi^2} d\xi\right)$, where $\eta = \frac{y}{2\sqrt{\nu t}}$. Then, at $t = T$, the plate stops. This is the same as adding an impulsive plate motion to the left to the already moving plate. Since the field equation is linear, the fluid motion can be obtained from such a superposition of impulsive events as well. Hence, for $t > T$,

$$u(y,t) = U\left(1 - \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\xi^2} d\xi\right) - U\left(1 - \frac{2}{\sqrt{\pi}}\int_0^{\eta_T} e^{-\xi^2} d\xi\right) = \frac{2U}{\sqrt{\pi}}\int_{\eta}^{\eta_T} e^{-\xi^2} d\xi$$

where $\eta$ is defined above and $\eta_T = \frac{y}{2\sqrt{\nu(t-T)}}$. Finding the height of the peak velocity means computing $\partial u/\partial y$, setting it equal to zero, and solving for $y$.

$$\frac{\partial u(y,t)}{\partial y} = \frac{2U}{\sqrt{\pi}}\left(e^{-\eta_T^2}\frac{\partial \eta_T}{\partial y} - e^{-\eta^2}\frac{\partial \eta_T}{\partial y}\right) = \frac{2U}{\sqrt{\pi}}\left(\frac{e^{-\eta_T^2}}{2\sqrt{\nu(t-T)}} - \frac{e^{-\eta^2}}{2\sqrt{\nu t}}\right) = 0$$

To find the vertical location of the peak velocity, divide out common factors, move one term to the other side of the equation, take the natural logarithm, and solve for $y$.

$$\frac{y^2}{4\nu(t-T)} + \ln\sqrt{t-T} = \frac{y^2}{4\nu t} + \ln\sqrt{t} \quad \text{->} \quad y = \left[\frac{2\nu(t-T)t}{T}\ln\left(\frac{t}{t-T}\right)\right]^{1/2}$$

b) $I = \int_0^T \tau_w dt = \mu\int_0^T \left(\frac{\partial u}{\partial y}\right)_{y=0} dt = -\mu\int_0^T \frac{U}{\sqrt{\pi\nu t}}\, dt = -\frac{2\mu U}{\sqrt{\pi\nu}} T^{1/2}$. The sign of this answer implies that the fluid impeded the motion of the plate.
c) The input mechanical impulse is dissipated at the surface of the plate. After the plate stops moving ($t > T$), the mechanical impulse extracted from the flow, $I_e(t)$, will be:

$$I_e(t) = \int_T^t \tau_w dt = \mu\frac{U}{\sqrt{\pi\nu}}\int_T^t \left(\frac{1}{\sqrt{t-T}} - \frac{1}{\sqrt{t}}\right) dt = \frac{2\mu U}{\sqrt{\pi\nu}}\left(\sqrt{t-T} - \sqrt{t} + \sqrt{T}\right)$$

Thus, $\frac{I_e(t)}{|I|} = 0.99 = \frac{\sqrt{t-T} - \sqrt{t} + \sqrt{T}}{T^{1/2}} = \sqrt{\frac{t}{T} - 1} - \sqrt{\frac{t}{T}} + 1$, or, assuming that $t >> T$,

$$0.01 = \sqrt{\frac{t}{T} - 1} - \sqrt{\frac{t}{T}} = \sqrt{\frac{t}{T}}\left(1 - \sqrt{1 - \frac{T}{t}}\right) \approx \sqrt{\frac{t}{T}}\left(1 - 1 + \frac{T}{2t} + \ldots\right) = \frac{1}{2}\sqrt{\frac{T}{t}} \quad \text{->} \quad t = \frac{T}{(0.02)^2} = 2500T.$$

Thus, the time for the fluid to slow down is very long time compared to the time that the plate was actually moving.

**Exercise 9.41**. Consider the development from rest of plane Couette flow. The flow is bounded by two rigid boundaries at $y = 0$ and $y = h$, and the motion is started from rest by suddenly accelerating the lower plate to a steady velocity $U$. The upper plate is held stationary. Here a similarity solution cannot exist because of the appearance of the parameter $h$. Show that the velocity distribution is given by

$$u(y,t) = U\left(1 - \frac{y}{h}\right) - \frac{2U}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp\left(-n^2\pi^2\frac{\nu t}{h^2}\right)\sin\left(\frac{n\pi y}{h}\right)$$

Sketch the flow pattern at various times, and observe how the velocity reaches the linear distribution for large times.

**Solution 9.41**. The differential equation is (9.20), $\dfrac{\partial u}{\partial t} = \nu\dfrac{\partial^2 u}{\partial y^2}$, and the boundary and initial conditions on $u(y,t)$ are: $u(0,t) = U$, $u(h,t) = 0$, and $u(y,0) = 0$. The solution is facilitated by defining $u(y,t) = U(1 - y/h) + q(y,t)$, where $q$ is the deviation of the velocity from the final linear distribution. The deviation velocity satisfies the same differential equation, but with homogeneous boundary conditions: $q(0,t) = q(h,t) = 0$, and the initial condition:

$$q(y,0) = U(1 - y/h).$$

A Fourier series solution for $q$ satisfying the two boundary conditions is:

$$q = \sum_{n=1}^{\infty} A_n \exp\left\{-n^2\pi^2\frac{\nu t}{h^2}\right\}\sin\left(n\pi y/h\right).$$

The values of the coefficients $A_n$ are found from the initial condition:

$$q(y,0) = U\left(1 - \frac{y}{h}\right) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{h}\right).$$

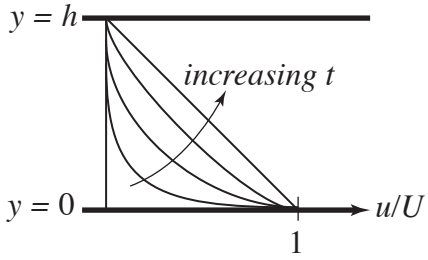
Utilizing the orthogonality of the sine-functions leads to:

$$A_n = \frac{2}{h}\int_0^h U\left(1 - \frac{y}{h}\right)\sin\left(\frac{n\pi y}{h}\right)dy = \frac{2U}{\pi n}.$$

Thus, the time-dependent velocity distribution is

$$u(y,t) = U\left(1 - \frac{y}{h}\right) - \frac{2U}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp\left(-n^2\pi^2\frac{\nu t}{h^2}\right)\sin\left(\frac{n\pi y}{h}\right).$$

The velocity profiles sketched below for different values of $\nu t/h^2$ show that the effect of the upper stationary boundary is initially negligible. Gradually it influences the entire region by viscous diffusion.

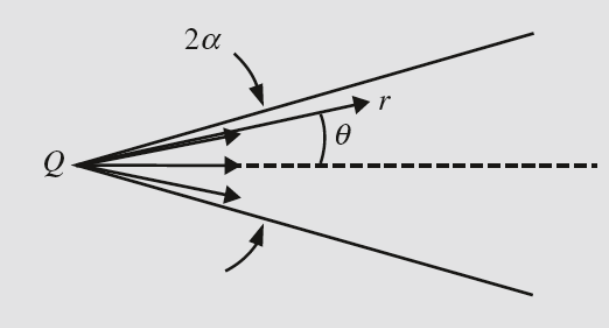


**Exercise 9.42**. Two-dimensional flow between flat non-parallel plates can be formulated in terms of a normalized angular coordinate, $\eta = \theta/\alpha$, where $\alpha$ is the half angle between the plates, and a normalized radial velocity, $u_r(r,\theta) = u_{\max}(r) f(\eta)$ where $\eta = \theta/\alpha$ for $|\theta| \le \alpha$. Here, $u_\theta = 0$, the Reynolds number is $Re = u_{\max} r\alpha/\nu$, and $Q$ is the volume flux (per unit width perpendicular to the page).

a) Using the appropriate versions of (4.10) and (8.1), show that $f''' + \mathrm{Re}\alpha f^2 + 4\alpha^2 f = const.$

b) Find $f(\eta)$ for symmetric creeping flow, i.e. $Re = 0 = f(+1) = f(-1)$, and $f(0) = 1$.

c) Above what value of the channel half-angle will backflow always occur?




**Solution 9.42**. a) In two-dimensional $(r,\theta)$-polar coordinates, the governing equations for steady radial flow, $u_r(r,\theta) = u_{\max}(r) f(\eta)$, are:

continuity: $$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) = 0,$$

radial momentum: $$\rho u_r \frac{\partial u_r}{\partial r} = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right) + \frac{1}{r^2}\frac{\partial u_r}{\partial \theta^2}\right], \text{ and}$$

$\theta$-direction momentum: $$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right].$$

Integrating the continuity equation once implies: $u_r = (\nu/r)F(\eta)$, where $\eta = \theta/\alpha$ and the factors of $\nu$ and $1/\alpha$ have been added so that the profile function $F(\eta)$ will be dimensionless and its boundary conditions can be applied at $\eta$ = ±1. Place this solution form into the momentum equations to find:

$$-\rho\frac{\nu^2 F^2}{r^3} = -\frac{\partial p}{\partial r} + \frac{\mu}{r^3}\frac{\nu}{\alpha^2}F'', \text{ and } 0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\frac{2\nu}{r^3}\frac{1}{\alpha}F'.$$

where a prime denotes $d/d\eta$. Simplify and isolate the pressure gradients:

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{\nu^2}{r^3}\left(F^2 + \frac{1}{\alpha^2}F''\right), \text{ and } \frac{1}{\rho}\frac{\partial p}{\partial \theta} = \frac{2\nu^2}{r^2}\frac{1}{\alpha}F'.$$

Cross differentiate these equations, eliminate the pressure, and remove common factors:

$$\frac{\nu^2}{r^3}\left(\frac{2}{\alpha}FF' + \frac{1}{\alpha^3}F'''\right) = -\frac{4\nu^2}{r^3}\frac{1}{\alpha}F', \text{ or } \frac{1}{\alpha^2}F''' + 2FF' + 4F' = 0.$$

The final equation can be integrated:

$$\frac{1}{\alpha^2}F'' + F^2 + 4F = const.$$

Now include the maximum velocity in the scaling by redefining the profile function:

$$f(\eta) \equiv \frac{F(\eta)}{F_{\max}} = \frac{u_r}{u_{\max}},$$

where $u_{\max} = (\nu/r)F_{\max}$. This introduces the Reynolds number based on the local half width ($\alpha r$) of the channel, $\text{Re} = \dfrac{u_{\max}\alpha r}{\nu} = \left(\dfrac{\nu}{r}F_{\max}\right)\dfrac{\alpha r}{\nu} = F_{\max}\alpha$, into the equation:

$$\frac{F_{\max}}{\alpha^2} f'' + F_{\max}^2 f^2 + 4F_{\max} f = const., \quad \text{or} \quad f'' + \alpha \,\text{Re}\, f^2 + 4\alpha^2 f = const.$$

b) For creeping flow set $Re = 0$, and this eliminates the nonlinear term: $f''' + 4\alpha^2 f' = 0$. The solutions for $f'$ will be: $f' = C_1\cos(2\alpha\eta) + C_2\sin(2\alpha\eta)$. These can be integrated once to find: $f = C_1\cos(2\alpha\eta) + C_2\sin(2\alpha\eta) + C_3$ where the unknown constants have been suitably redefined. The constants can be evaluated using the boundary conditions:

$$f(+1) = 0 = C_1\cos(2\alpha) + C_2\sin(2\alpha) + C_3$$
$$f(-1) = 0 = C_1\cos(2\alpha) - C_2\sin(2\alpha) + C_3$$
$$f(0) = 1 = C_1 + C_3$$

Here, $C_1 = (1/2)\csc^2\alpha$, $C_2 = 0$, and $C_3 = 1 - (1/2)\csc^2\alpha$, so that:

$f(\eta) = 1 + \dfrac{1}{2}\csc^2(\alpha)\left[\cos(2\alpha\eta) - 1\right]$, or with a little more manipulation: $f(\eta) = 1 - \dfrac{\sin^2(\alpha\eta)}{\sin^2(\alpha)}$

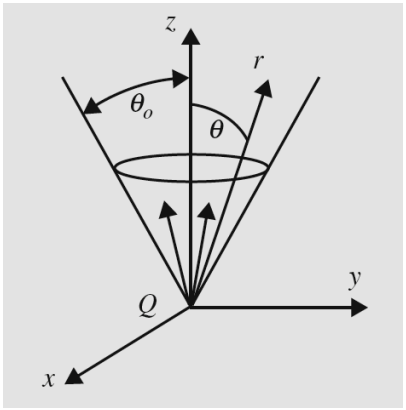
c) For backflow to occur, $f$ must be negative. This will occur if $\sin^2(\alpha\eta) > \sin^2(\alpha)$, which can happen if $\alpha > \pi/2$.

**Exercise 9.43**. Consider steady viscous flow inside a cone of constant angle $\theta_o$. The flow has constant volume flux = $Q$, and the fluid has constant density = $\rho$ and constant kinematic viscosity = $\nu$. Use spherical coordinates, and assume that the flow only has a radial component, $\mathbf{u} = \left(u_r(r,\theta),0,0\right)$ which is independent of the azimuthal angle $\varphi$, so that the equations of motion are:

Cons. of Mass: $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) = 0$$

Cons. of Radial Mom.: $$u_r\frac{\partial u_r}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_r}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_r}{\partial\theta}\right) - \frac{2}{r^2}u_r\right)$$

Cons. of $\theta$-Mom. $$0 = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + \nu\left(\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}\right)$$




For the following items, assume the radial velocity can be determined using: $u_r(r,\theta) = QR(r)\Theta(\theta)$. Define the Reynolds number of this flow as: $Re = Q/(\pi\nu r)$.

a) Use the continuity equation to determine $R(r)$.

b) Integrate the $\theta$-momentum equation, assume the constant of integration is zero, and combine the result with the radial mom. equ. to determine a single differential equation for $\Theta(\theta)$ in terms of $\theta$ and $Re$. It is not necessary to solve this equation.

c) State the matching and/or boundary conditions that $\Theta(\theta)$ must satisfy.

**Solution 9.43**. a) Place $u_r(r,\theta) = QR(r)\Theta(\theta)$ into $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) = 0$, to find: $\frac{\partial}{\partial r}\left(r^2 R\right) = 0$ for $r \neq 0$.

Thus: $r^2R$ = constant or $R \propto 1/r^2$, so $u_r(r,\theta) = Qr^{-2}\Theta(\theta)$, so $\Theta$ is a dimensionless function.

b) Rewrite the $\theta$-mom. equ.: $\frac{1}{\rho}\frac{\partial p}{\partial\theta} = \nu\left(\frac{2}{r}\frac{\partial u_r}{\partial\theta}\right)$, integrate with respect to $\theta$: $\frac{p}{\rho} = \frac{2\nu}{r}u_r + f(r)$.

Now set $f = 0$, and differentiate with respect to $r$ to get: $\frac{1}{\rho}\frac{\partial p}{\partial r} = 2\nu\frac{\partial}{\partial r}\left(\frac{u_r}{r}\right)$. Use this in the $r$-mom. equ. to eliminate the pressure:

$$u_r\frac{\partial u_r}{\partial r} = -2\nu\frac{\partial}{\partial r}\left(\frac{u_r}{r}\right) + \nu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_r}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_r}{\partial\theta}\right) - \frac{2}{r^2}u_r\right)$$

Now plug in $u_r(r,\theta) = Qr^{-2}\Theta(\theta)$ and work to find the differential equation for $\Theta$.

$$\frac{Q\Theta}{r^2}\frac{\partial}{\partial r}\left(\frac{Q\Theta}{r^2}\right) = -2\nu\frac{\partial}{\partial r}\left(\frac{Q\Theta}{r^3}\right) + \nu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{-2Q\Theta}{r}\right) + \frac{Q}{r^4\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{2Q\Theta}{r^4}\right)$$

Perform derivatives, cancel opposite terms, divide out the common factor of $Q/r^4$, and continue simplifying.

$$-2\frac{Q^2\Theta^2}{r^5} = +6\nu\frac{Q\Theta}{r^4} + \nu\left(\frac{2Q\Theta}{r^4} + \frac{Q}{r^4\sin\theta}\left(\cos\theta\frac{d\Theta}{d\theta} + \sin\theta\frac{d^2\Theta}{d\theta^2}\right) - \frac{2Q\Theta}{r^4}\right)$$

$$-2\frac{Q\Theta^2}{\nu r} = 6\Theta + \cot\theta\frac{d\Theta}{d\theta} + \frac{d^2\Theta}{d\theta^2}$$

Plug in the Reynolds number definition and rewrite the equation in the ordinary order:

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta \frac{d\Theta}{d\theta} + 6\Theta + 2\pi Re\Theta^2 = 0$$

c) The radial flow speed must be zero at the angle defined by the surface of the cone. Thus, the first boundary condition is: $\Theta(\theta_o) = 0$. The second matching condition comes from the requirement that that the volume flux be equal to $Q$.

$$Q = 2\pi \int_0^{\theta_o} u_r(r,\theta) r^2 \sin\theta d\theta = 2\pi \int_0^{\theta_o} Q\Theta(\theta) \sin\theta d\theta, \quad \text{or} \quad \frac{1}{2\pi} = \int_0^{\theta_o} \Theta(\theta) \sin\theta d\theta$$

These two conditions will be sufficient for determining the unknown constants that will arise from integrating the second order differential equation found for part b). In addition there is also a smoothness constraint, $\left(d\Theta/d\theta\right)_{\theta=0} = 0$, which may be needed to prevent the flow profile from having a conical peak at $\theta = 0$, and the requirement that $\Theta$ is finite for $0 \le \theta \le \theta_o$.

**Exercise 9.44**. The boundary conditions on obstacles in Hele-Shaw flow were not considered in Example 9.4. Therefore, consider them here by examining Hele-Shaw flow parallel to a flat obstacle surface at $y = 0$. The Hele-Shaw potential in this case is:

$$\phi = Ux\frac{z}{h}\left(1-\frac{z}{h}\right),$$

where $(x, y, z)$ are Cartesian coordinates and the flow is confined to $0 < z < h$ and $y > 0$.
a) Show that this potential leads to a slip velocity of $u(x, y \to 0) = U\left(z/h\right)\left(1 - z/h\right)$, and determine the pressure distribution implied by this potential.
b) Since this is a viscous flow, the slip velocity must be corrected to match the genuine no-slip condition on the obstacle's surface at $y = 0$. The analysis of Example (9.4) did not contain the correct scaling for this situation near $y = 0$. Therefore, rescale the $x$-component of (9.1) using: $x^* = x/L$, $y^* = y/h = y/\varepsilon L$, $z^* = z/h = z/\varepsilon L$, $t^* = Ut/L$, $u^* = u/U$, $v^* = v/\varepsilon U$, $w^* = w/\varepsilon U$, and $p^* = p/P_a$, and then take the limit as $\varepsilon^2 \mathrm{Re}_L \to 0$, with $\mu UL/P_a h^2$ remaining of order unity, to simplify the resulting dimensionless equation that has

$$0 \cong \frac{dp}{dx} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

as its dimensional counterpart.
c) Using boundary conditions of $u = 0$ on $y = 0$, and $u = U\left(z/h\right)\left(1 - z/h\right)$ for $y >> h$. Show that

$$u(x,y,z) = U\frac{z}{h}\left(1-\frac{z}{h}\right) + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{h}z\right)\exp\left(-\frac{n\pi}{h}y\right), \text{ where } A_n = -\frac{2U}{h}\int_0^h \frac{z}{h}\left(1-\frac{z}{h}\right)\sin\left(\frac{n\pi}{h}z\right)dz,$$

which implies: $A_n = -8U/n^3\pi^3$ for $n$ = odd, and $A_n = 0$ for $n$ = even. [The results here are directly applicable to the surfaces of curved obstacles in Hele-Shaw flow when the obstacle's radius of curvature is much greater than $h$.]

**Solution 9.44**. a) From Example 9.4, the $u$ and $v$ velocity components implied by the given potential are:

$$u = \frac{\partial}{\partial x}\phi = U\frac{z}{h}\left(1-\frac{z}{h}\right) \text{ and } v = \frac{\partial}{\partial y}\phi = 0.$$

Neither of these equations shows any dependence on $y$, so, as $y \to 0$, the velocity components are as given above and there is slip at the obstacle surface $y = 0$. Example 8.2 also shows that

$$\phi = -\frac{z(h-z)}{2\mu}p = Ux\frac{z}{h}\left(1-\frac{z}{h}\right), \text{ so } p = -\frac{2\mu U}{h^2}x.$$

b) The $x$-component of (9.1) is:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right).$$

Insert the given scaling into this equation, to find:

$$\frac{U}{L/U}\frac{\partial u^*}{\partial t^*} + \frac{U^2}{L}u^*\frac{\partial u^*}{\partial x^*} + \frac{\varepsilon U^2}{h}v^*\frac{\partial u^*}{\partial y^*} + \frac{\varepsilon U^2}{h}w^*\frac{\partial u^*}{\partial z^*} = -\frac{P_a}{\rho L}\frac{\partial p^*}{\partial x^*} + \nu\left(\frac{U}{L^2}\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{U}{h^2}\frac{\partial^2 u^*}{\partial y^{*2}} + \frac{U}{h^2}\frac{\partial^2 u^*}{\partial z^{*2}}\right),$$

Simplify using $\varepsilon = h/L$, and collect terms with like coefficients:

$$\frac{U^2}{L}\left(\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}+w^*\frac{\partial u^*}{\partial z^*}\right)=-\frac{P_a}{\rho L}\frac{\partial p^*}{\partial x^*}+\nu\frac{U}{L^2}\frac{\partial^2 u^*}{\partial x^{*2}}+\nu\frac{U}{h^2}\left(\frac{\partial^2 u^*}{\partial y^{*2}}+\frac{\partial^2 u^*}{\partial z^{*2}}\right).$$

Multiply through by $\rho h^2/\mu U$, and use $\varepsilon = h/L$ and $\mathrm{Re}_L = \rho UL/\mu$ to reach:

$$\varepsilon^2\,\mathrm{Re}_L\left(\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}+w^*\frac{\partial u^*}{\partial z^*}\right)=-\frac{P_a h^2}{\mu UL}\frac{\partial p^*}{\partial x^*}+\varepsilon^2\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{\partial^2 u^*}{\partial y^{*2}}+\frac{\partial^2 u^*}{\partial z^{*2}}.$$

When $\varepsilon^2\mathrm{Re}_L \to 0$, but $\mu UL/P_a h^2$ remains of order unity this equation simplifies to:

$$0 \cong -\frac{P_a h^2}{\mu UL}\frac{\partial p^*}{\partial x^*}+\frac{\partial^2 u^*}{\partial y^{*2}}+\frac{\partial^2 u^*}{\partial z^{*2}}, \text{ or in dimensional form: } 0 \cong -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right).$$

c) The field equation derived in part b) for u is linear. Therefore assume that it can be written as a sum of two terms, one that provides the correct behavior when $y >> h$ and one that acts to provide the correct boundary condition at $y = 0$:

$$u = U\frac{z}{h}\left(1-\frac{z}{h}\right)+\upsilon(y,z), \tag{i}$$

where $\upsilon = 0$ when $z = 0$ and $z = h$, $\upsilon \to 0$ when $y >> h$, and $u \to 0$ when $y \to 0$. Using $-\partial p/\partial x = 2\mu U/h^2$, the part b) field equation becomes:

$$\frac{1}{\rho}\frac{2\mu U}{h^2}+\nu\left(-\frac{2U}{h^2}\right)+\nu\left(\frac{\partial^2\upsilon}{\partial y^2}+\frac{\partial^2\upsilon}{\partial z^2}\right)=0, \text{ or } \frac{\partial^2\upsilon}{\partial y^2}+\frac{\partial^2\upsilon}{\partial z^2}=0.$$

Thus, $\upsilon$ solves a two dimensional Laplace equation. Here, a separation of variable solution, $\upsilon = Y(y)Z(z)$ is appropriate. Placing this solution form into the two dimensional Laplace equation and dividing by $\upsilon$, produces:

$$\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=0.$$

The first term depends only on $y$ and the second term depends only on $z$, so each must be constant for the equation to hold throughout the domain. Let the first term equal $+k^2$ and the second term $= -k^2$, so that the individual Y, and Z equations become:

$$\frac{d^2Y}{dy^2}-k^2Y=0, \text{ and } \frac{d^2Z}{dz^2}+k^2Z=0.$$

The solutions to these two equations are: $Y(y) = A_\pm \exp\{\pm ky\}$, and $Z(z) = B\sin(kz)+C\cos(kz)$.

To satisfy the boundary conditions $\upsilon = 0$ when $z = 0$ and $z = h$, implies $C = 0$ and $B\sin(kh) = 0$. The second of these is satisfied by the (useless) trivial solution $B = 0$, or by $\sin(kh) = 0$. The later condition leads to $k = k_n = n\pi/h$, where $n$ is a positive integer so that $Z(z) = B\sin(n\pi z/h)$. Thus, the separation constant k may take on discrete values.

To satisfy the boundary condition $\upsilon \to 0$ when $y >> h$, means setting $A_+ = 0$. Therefore, $\upsilon(y, z)$ takes the following form:

$$\upsilon(y,z)=\sum_{n=1}^{\infty}A_n\sin(k_n z)\exp\{-k_n y\}. \tag{ii}$$

to allow for all possible values of $k$. To satisfy the final boundary condition ( $u \to 0$ when $y \to 0$ ), (i) and (ii) imply:

$$U\frac{z}{h}\left(1-\frac{z}{h}\right)+\upsilon(0,z)=0=U\frac{z}{h}\left(1-\frac{z}{h}\right)+\sum_{n=1}^{\infty}A_n\sin(k_n z),$$

and this equation can be used to determine the coefficients $A_n$. Following the usual approach for a Fourier series, multiply it by $\sin(k_m z)$ and integrate in $z$ from 0 to $h$. The result is:

$$\int_0^h \sin(k_m z) U \frac{z}{h}\left(1-\frac{z}{h}\right)dz = -\sum_{n=1}^{\infty} A_n \int_0^h \sin(k_m z)\sin\left(k_n z\right)dz = -\sum_{n=1}^{\infty} A_n \frac{h}{2}\delta_{mn},$$

where $\delta_{mn}$ is the Kronecker delta function, so the $A_m$'s are given by:

$$A_m = -\frac{2U}{h}\int_0^h \sin(k_m z)\frac{z}{h}\left(1-\frac{z}{h}\right)dz. \qquad \text{(iii)}$$

Thus, the final velocity field is:

$$u(y,z) = U\frac{z}{h}\left(1-\frac{z}{h}\right) + \sum_{m=1}^{\infty} A_n \sin\left(\frac{m\pi}{h}z\right)\exp\left\{-\frac{m\pi}{h}y\right\}$$

where $A_n$ is given by (iii). The integral is tedious but can be evaluated by parts to find: $A_n = -8U/n^3\pi^3$ for $n$ = odd, and $A_n = 0$ for $n$ = even.

The final answer includes a sum of terms that decay exponentially with increasing $y$, and the first of these will have the slowest decay. At a distance of $y = h$, this slowest decaying term will be a factor of $e^{-\pi} \approx 0.04$ smaller than it is at $y = 0$.

**Exercise 9.45**. Using the velocity field (8.49), determine the drag on Stokes' sphere from the surface pressure and the viscous surface stresses $\sigma_{rr}$ and $\sigma_{r\theta}$.

**Solution 9.45**. There are pressure and shear stress contributions to the drag on a moving sphere at low Reynolds number. The pressure distribution is given by (8.50):

$$p(r,\theta) - p_\infty = -\frac{3\mu a U}{2r^2}\cos\theta.$$

The pressure drag can be obtained by integrating this result:

$$F_{pressure} = -\int_{surface} p(r=a,\theta)\mathbf{e}_r \cdot \mathbf{e}_z dS = -2\pi a^2 \int_{\theta=0}^{\theta=\pi} \mu\left(\frac{3U}{2a}\right)\cos^2\theta\sin\theta d\theta = 2\pi\mu U a$$

The viscous drag can be obtained from surface integrals of the viscous stresses:

$$F_{vicous} = -\int_{surface} \sigma_{r\theta}(r=a,\theta)\mathbf{e}_\theta \cdot \mathbf{e}_z dS + \int_{surface} \sigma_{rr}(r=a,\theta)\mathbf{e}_r \cdot \mathbf{e}_z dS$$

$$= -2\pi a^2 \int_{\theta=0}^{\theta=\pi} \sigma_{r\theta}(r=a,\theta)\sin^2\theta d\theta + 2\pi a^2 \int_{\theta=0}^{\theta=\pi} \sigma_{rr}(r=a,\theta)\cos\theta\sin\theta d\theta,$$

where $\sigma_{r\theta} = \mu\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) = -\frac{\mu U\sin\theta}{r}\left(\frac{3a^3}{2r^3}\right)$, and $\sigma_{rr} = 2\mu\frac{\partial u_r}{\partial r} = 2\mu U\cos\theta\left(\frac{3a}{2r^2} - \frac{3a^3}{2r^4}\right)$.

Thus, at $r = a$, $\sigma_{r\theta} \neq 0$, but $\sigma_{rr} = 0$, so

$$F_{vicous} = 3\pi\mu U a \int_{\theta=0}^{\theta=\pi} \sin^3\theta d\theta = 4\pi\mu U a.$$

Thus, one third of the drag comes from pressure forces and two thirds come from the shear stress. The total drag is the sum of these two contributions:

$$F_{drag} = F_{pressure} + F_{vicous} = 2\pi\mu U a + 4\pi\mu U a = 6\pi\mu U a.$$

**Exercise 9.46**. Calculate the drag on a spherical droplet of radius $r = a$, density $\rho'$ and viscosity $\mu'$ moving with velocity $U$ in an infinite fluid of density $\rho$ and viscosity $\mu$. Assume Re = $\rho Ua/\mu$ << 1. Neglect surface tension.

**Solution 9.46**. The effort here is similar to that in Section 8.6 for low Reynolds number flow past a sphere. The main difference being that flow inside and outside the sphere must be considered simultaneously. In the following solution, dimensionless variables are used without any special notation. These are: $u_r/U$, $u_\theta/U$, $r/a$, $\psi/(Ua^2)$, $(p - p_\infty)/(\mu U/a)$, and $\tau/(\mu U/a)$, so that the surface of the sphere is $r = 1$, and the free stream speed is unity. The development leading to (8.48) suggests that the flow *outside* the sphere can be determined from:

$$\psi_o = \left(A_o r^4 + B_o r^2 + C_o r + D_o/r\right)\sin^2\theta \quad \text{for} \quad r > a.$$

To recover a uniform flow far from the sphere, the first two constants must be $A = 0$ and $B = 1/2$.

Similarly, the flow inside the sphere can be represented by

$$\psi_i = \left(A_i r^4 + B_i r^2 + C_i r + D_i/r\right)\sin^2\theta \quad \text{for} \quad r < a.$$

Here, $D_i = 0$ to avoid a singularity at $r = 0$.

Compute the velocities inside and outside the sphere:

$$u_r = -\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} = \begin{Bmatrix} -2\cos\theta\left(\dfrac{1}{2} + \dfrac{C_o}{r} + \dfrac{D_o}{r^3}\right) & \text{for } r > 1 \\ -2\cos\theta\left(A_i r^2 + B_i + \dfrac{C_i}{r}\right) & \text{for } r < 1 \end{Bmatrix}, \text{ and}$$

$$u_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} = \begin{Bmatrix} \sin\theta\left(1 + \dfrac{C_o}{r} - \dfrac{D_o}{r^3}\right) & \text{for } r > 1 \\ \sin\theta\left(4A_i r^2 + 2B_i + \dfrac{C_i}{r}\right) & \text{for } r < 1 \end{Bmatrix}.$$

For a bounded velocity inside the sphere, $C_i$ must be 0. There are four remaining boundary and matching conditions at $r = 1$:

$$u_r(1^+) = 0 \text{ and } u_r(1^-) = 0, \text{ and these imply } \frac{1}{2} + C_o + D_o = 0, \text{ and } A_i + B_i = 0; \qquad (1,2)$$

$$u_\theta(1^+) = u_\theta(1^-), \text{ and this implies } 1 + C_o - D_o = 4A_i + 2B_i; \text{ and} \qquad (3)$$

the stress continuity condition:

$$\sigma_{r\theta}(a^+) = \sigma_{r\theta}(a^-) \text{ or } \mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right]_{r=1^+} = \mu'\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right]_{r=1^-}, \text{ and this implies}$$

$$\mu\sin\theta\left[-1 - 2C_o + 4D_o + 2\frac{1}{2} + 2C_o + 2D_o\right] = \mu'\sin\theta\left[4A_i - 2B_i + 2A_i + 22B_i\right].$$

Fortunately, there are many canceling terms in this last condition and it simplifies to:

$$6\mu D_o = 6\mu' A_i a. \qquad (4)$$

Simultaneous solution of the four algebraic equations produces:

$$C_o = -\frac{1}{4}\frac{3\mu' + 2\mu}{\mu' + \mu}, \quad D_o = \frac{1}{4}\frac{\mu'}{\mu' + \mu}, \quad A_i = \frac{1}{4}\frac{\mu}{\mu' + \mu}, \text{ and } \quad B_i = -\frac{1}{4}\frac{\mu}{\mu' + \mu}.$$

The force on the sphere is determined from a surface intergral:

$$F_i = -\int_{\text{sphere's surface}} \left(p\delta_{ij} - \sigma_{ij}\right)n_j\, dA.$$

In the following, force is made dimensionless by $\mu Ua$, area is made dimensionless with $a^2$, and stress is made dimensionless with $\mu U/a$. The normal and surface elements are:

$$\mathbf{n} = \mathbf{e}_r, \text{ and } dA = \left[r^2 \sin\theta d\theta d\varphi\right]_{r=1}.$$

The dot product of the shear stress and the directed area element is:

$$\tau_{ij} n_j dA = \left\{ \mathbf{e}_r 2\frac{\partial u_r}{\partial r} + \mathbf{e}_\theta \left( \frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) \right\} \sin\theta d\theta d\varphi.$$

The $\theta$-component of surface shear stress

$$\mathbf{e}_\theta 6(D_o/a^3)\sin^2\theta d\theta d\varphi = \mathbf{e}_\theta \frac{3}{2}\frac{\mu'}{\mu+\mu'}\sin\theta d\theta d\varphi.$$

The r-component includes a pressure contribution which must be found by integrating Stokes' creeping-flow momentum equation:

$$\psi_o = \left( \frac{r^2}{2} + \frac{1}{4}\frac{\mu'}{\mu'+\mu}\frac{1}{r} - \frac{1}{4}\frac{3\mu'+2\mu}{\mu'+\mu} r \right)\sin^2\theta, \text{ and } -\nabla p = \nabla\times\nabla\times\mathbf{u} = -\nabla^2\mathbf{u}.$$

In this dimensionless formulation, these give:

$$-\frac{\partial p}{\partial r} = \frac{1}{r^2\sin\theta}\left[ \sin\theta\cos\theta\left(1 + \frac{1}{2}\frac{\mu'}{\mu'+\mu}\frac{1}{r^3}\right) - \left(\frac{1}{2} + \frac{1}{4}\frac{\mu'}{\mu'+\mu}\frac{1}{r^3} - \frac{1}{4}\frac{3\mu'+2\mu}{\mu'+\mu}\frac{1}{r}\right)4\sin\theta\cos\theta \right]$$

$$= \frac{3\mu'+2\mu}{\mu'+\mu}\frac{\cos\theta}{r^3}$$

$$-\frac{1}{r}\frac{\partial p}{\partial\theta} = \frac{\sin^2\theta}{r\sin\theta}\left(-\frac{3}{2}\frac{\mu'}{\mu'+\mu}\frac{1}{r^4}\right) - \frac{2\sin\theta}{r}\left[\frac{\partial}{\partial r}\left(\frac{1}{2} + \frac{1}{4}\frac{\mu'}{\mu'+\mu}\frac{1}{r^3} - \frac{1}{4}\frac{3\mu'+2\mu}{\mu'+\mu}\frac{1}{r}\right)\right], \text{ or}$$

$$-\frac{\partial p}{\partial\theta} = \frac{3\mu'+2\mu}{\mu'+\mu}\frac{\sin\theta}{2r^2}.$$

Integrating for $p$ and setting $p_\infty = 0$ produces:

$$p = \frac{3\mu'+2\mu}{\mu'+\mu}\frac{\cos\theta}{2r^2}.$$

Thus, the $r$-component of the force integrand is:

$$\left[\left(-p + 2\frac{\partial u_r}{\partial r}\right)\mathbf{e}_r\left[r^2\sin\theta d\theta d\varphi\right]_{r=1}\right] = -\frac{3}{2}\cos\theta\frac{\mu'+2\mu}{\mu'+\mu}\mathbf{e}_r\sin\theta d\theta d\varphi.$$

So, the force on the liquid sphere is:

$$\mathbf{F} = \int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\left[-\mathbf{e}_r\frac{3}{2}\cos\theta\frac{\mu'+2\mu}{\mu'+\mu} + \mathbf{e}_\theta\frac{3}{2}\frac{\mu'}{\mu+\mu'}\right]\sin\theta d\theta d\varphi.$$

The unit vectors on the surface of the sphere are functions of the polar and azimuthal angles, $\theta$, $\varphi$, respectively. They must be put in terms of the Cartesian unit vectors that do not vary with position. The necessary relationships are:

$$\mathbf{e}_r = \mathbf{e}_x\cos\varphi\sin\theta + \mathbf{e}_y\sin\varphi\sin\theta + \mathbf{e}_z\cos\theta, \text{ and } \mathbf{e}_\theta = \mathbf{e}_x\cos\varphi\cos\theta + \mathbf{e}_y\sin\varphi\cos\theta - \mathbf{e}_z\sin\theta.$$

Integrating first over j, the $x$- and $y$-direction contributions to the force are zero, a result that can also be deduced from the axial symmetry of the flow. This leaves:

$$\mathbf{F} = 2\pi\mathbf{e}_z\int_{\theta=0}^{\pi} -\left[\frac{3}{2}\frac{\mu'+2\mu}{\mu'+\mu}\cos^2\theta\sin\theta - \frac{3}{2}\frac{\mu'}{\mu+\mu'}\sin^3\theta\right]d\theta.$$

Carrying out the trigonometric integrals leads to:

$$\mathbf{F} = -2\pi\mathbf{e}_z\left[\frac{\mu' + 2\mu}{\mu' + \mu} + \frac{2\mu'}{\mu + \mu'}\right] = -2\pi\mathbf{e}_z\,\frac{3\mu' + 2\mu}{\mu' + \mu}.$$

The force on the sphere is a drag force, and in dimensionless form this drag becomes:

$$D = 2\pi\mu U a\frac{3\mu' + 2\mu}{\mu' + \mu} = D_s\,\frac{3\mu' + 2\mu}{3\mu' + 3\mu}.$$

where $D_s = 6\pi\mu U a$ is the Stokes' drag on a solid sphere.

**Exercise 9.47**. Consider a very low Reynolds number flow over a circular cylinder of radius $r = a$. For $r/a = O(1)$ in the $\mathrm{Re} = Ua/\nu \to 0$ limit, find the equation governing the stream function $\psi(r, \theta)$ and solve for $\psi$ with the least singular behavior for large $r$. There will be one remaining constant of integration to be determined by asymptotic matching with the large $r$ solution (which is not part of this problem). Find the domain of validity of your solution.

**Solution 9.47**. The Stokes' flow field equation is:

$$\nabla p = \mu \nabla^2 \mathbf{u},$$

which can be rewritten:

$$\nabla p = -\mu \nabla \times \nabla \times \mathbf{u}.$$

Applying one more curl, eliminates the pressure gradient:

$$0 = \nabla \times \nabla \times \nabla \times \mathbf{u}.$$

In two-dimensional $(r,\theta)$-polar coordinates, the relationship between the stream function $\psi(r,\theta)$ and the velocity field is $\mathbf{u} = -\mathbf{e}_z \times \nabla \psi$, so the field equation for the stream function is given by:

$$0 = \nabla \times \nabla \times \nabla \times \left(-\mathbf{e}_z \times \nabla \psi\right).$$

This relationship can be converted to coordinate specific derivatives in steps. Start with the stream function definition.

$$-\mathbf{e}_z \times \nabla \psi = \frac{1}{r}\frac{\partial \psi}{\partial \theta}\mathbf{e}_r - \frac{\partial \psi}{\partial r}\mathbf{e}_\theta$$

Apply the first curl operation, using the formulae in Appendix B:

$$\nabla \times \left(-\mathbf{e}_z \times \nabla \psi\right) = -\mathbf{e}_z \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}\right].$$

Apply the second curl operation,

$$\nabla \times \nabla \times \left(-\mathbf{e}_z \times \nabla \psi\right) = -\mathbf{e}_r \frac{1}{r}\frac{\partial}{\partial \theta}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}\right] + \mathbf{e}_\theta \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}\right].$$

Apply third and final curl operation,

$$0 = \nabla \times \nabla \times \nabla \times \left(-\mathbf{e}_z \times \nabla \psi\right) =$$

$$\mathbf{e}_z \left\{\frac{1}{r}\frac{\partial}{\partial r} r\left(\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}\right]\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta^2}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}\right]\right\}.$$

For large $r$, $u_r \sim \cos\theta$, and $u_\theta \sim -\sin\theta$, so it is expected that $\psi \sim \sin\theta$. Therefore, a trial solution in the form $\psi(r,\theta) = f(r)\sin\theta$ may be sufficient. The resulting equation for $f$ looks like:

$$\frac{1}{r}\frac{\partial}{\partial r} r\left(\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) - \frac{f}{r^2}\sin\theta\right]\right) - \frac{\sin\theta}{r^3}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{f}{r^4}\sin\theta = 0.$$

Divide by $\sin\theta$ and recognize that this equation is equi-dimensional in $r$ so it has power-law solutions, $f(r) \sim (r/a)^m$, where the factor of a has been added for later scaling analysis. Substituting this into the equation for $f(r)$ and dividing by $(r/a)^{m-4}$ leaves:

$$m^2(m-2)^2 - (m-2)^2 - m^2 + 1 = 0.$$

This can be factored:

$$(m-1)^2(m+1)(m-3) = 0.$$

Thus, $m = 1$ is a double root which implies that the solutions for $f$ may be proportional to $r/a$ and $(r/a)\ln(r/a)$. Therefore:

$$f(r) = c_1(r/a) + c_2(r/a)\ln(r/a) + c_3(r/a)^3 + c_4(r/a)^{-1}.$$

The growth of $r^3$ is too fast to match the flow at large $r$, so $c_3$ must be zero. This leaves:

$$\psi(r,\theta) = \left(c_1(r/a) + c_2(r/a)\ln(r/a) + c_4(r/a)^{-1}\right)\sin\theta.$$

The boundary conditions at $r = a$ are:

$$u_r = \left[\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right]_{r=a} = (c_1 + c_4)\cos\theta = 0 \quad \text{or} \quad c_4 = -c_1, \text{ and}$$

$$u_\theta = \left[-\frac{\partial\psi}{\partial r}\right]_{r=a} = \frac{1}{a}(c_1 + c_2 - c_4)\sin\theta = 0 \quad \text{or } c_2 = -2c_1.$$

Therefore:

$$\psi(r,\theta) = c_1\left(\frac{r}{a} - 2\frac{r}{a}\ln\frac{r}{a} - \frac{a}{r}\right)\sin\theta.$$

The final constant is determined by matching this solution to the free-stream at large $r/a$. However, for large $r/a$, $\psi \sim (r/a)\ln(r/a)\sin\theta$ so $u_r \sim \ln(r/a)\cos\theta$ and $u_\theta \sim \ln(r/a)\sin\theta$. The advective acceleration $\mathbf{u}\cdot\nabla\mathbf{u} \sim (a/r)\ln(r/a)$ and the viscous stresses $\sim \nabla^2 u \sim (a/r)^2$. The terms neglected ~ terms retained when $\mathrm{Re}(a/r)\ln(r/a) \sim (a/r)^2$ or $(r/a)\ln(r/a) \sim 1/\mathrm{Re}$. So, this solution is valid for $(r/a)\ln(r/a) << 1/\mathrm{Re}$.

**Exercise 9.48**. A small neutrally-buoyant sphere is centered at the origin of coordinates in a deep bath of a quiescent viscous fluid with density $\rho$ and viscosity $\mu$. The sphere has radius $a$ and is initially at rest. It begins rotating about the $z$-axis with a constant angular velocity $\Omega$ at $t = 0$. The relevant equations for the fluid velocity, $\mathbf{u} = (u_r, u_\theta, u_\varphi)$, in spherical coordinates $(r,\theta,\varphi)$ are:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(u_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(u_\varphi\right) = 0 \ , \text{ and}$$

$$\frac{\partial u_\varphi}{\partial t} + u_r\frac{\partial u_\varphi}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\varphi}{\partial\theta} + \frac{u_\varphi}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi} + \frac{1}{r}\left(u_r u_\varphi + u_\theta u_\varphi \cot\theta\right) = -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\varphi} +$$

$$\nu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_\varphi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\varphi}{\partial\varphi^2} - \frac{u_\varphi}{r^2\sin^2\theta} + \frac{2}{r^2\sin^2\theta}\frac{\partial u_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\varphi}\right).$$

a) Assume $\mathbf{u} = \left(0,0,u_\varphi\right)$ and reduce these equations to:

$$\frac{\partial u_\varphi}{\partial t} = -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\varphi} + \nu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_\varphi}{\partial\theta}\right) - \frac{u_\varphi}{r^2\sin^2\theta}\right)$$

b) Set $u_\varphi(r,\theta,t) = \Omega a F(r,t)\sin\theta$, make an appropriate assumption about the pressure field, and derive the following equation for $F$: $\dfrac{\partial F}{\partial t} = \nu\left(\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial F}{\partial r}\right) - 2\dfrac{F}{r^2}\right)$.

c) Determine $F$ for $t \to \infty$ for boundary conditions $F = 1$ at $r = a$, and $F \to 0$ as $r \to \infty$.

d) Find the surface shear stress and torque on the sphere

**Solution 9.48**. ) When $u_r$ and $u_\theta$ are set equal to zero, the equations become: $\dfrac{\partial}{\partial\varphi}\left(u_\varphi\right) = 0$, and

$$\frac{\partial u_\varphi}{\partial t} + \frac{u_\varphi}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi} = -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\varphi} +$$

$$\nu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_\varphi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\varphi}{\partial\varphi^2} - \frac{u_\varphi}{r^2\sin^2\theta} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\varphi}{\partial\varphi}\right).$$

The remnant of the continuity equation, $\dfrac{\partial}{\partial\varphi}\left(u_\varphi\right) = 0$, implies that $u_\varphi$ does not depend on $\varphi$. Thus, three terms can be dropped from the second equation, so it simplifies to:

$$\frac{\partial u_\varphi}{\partial t} = -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\varphi} + \nu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_\varphi}{\partial\theta}\right) - \frac{u_\varphi}{r^2\sin^2\theta}\right).$$

b) The symmetry of the flow suggests that $p$ should not depend on the $\varphi$. Thus, setting $\partial p/\partial\varphi = 0$ and $u_\varphi(r,\theta,t) = \Omega a F(r,t)\sin\theta$ in the result of part a) produces:

$$\Omega a\sin\theta\frac{\partial F}{\partial t} = \nu\left(\frac{\Omega a\sin\theta}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right) + \frac{\Omega a F}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\sin\theta}{\partial\theta}\right) - \frac{\Omega a F\sin\theta}{r^2\sin^2\theta}\right).$$

Cancel the common factor of $\Omega a$, simplify terms, and perform the inner $\theta$-derivative:

$$\sin\theta\frac{\partial F}{\partial t} = \nu\left(\frac{\sin\theta}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right) + \frac{F}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\cos\theta\right) - \frac{F}{r^2\sin\theta}\right)$$

Perform the second $\theta$-derivative and note that $\cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$:

$$\sin\theta\frac{\partial F}{\partial t}=\nu\left(\frac{\sin\theta}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right)+\frac{F}{r^2\sin\theta}\left(1-2\sin^2\theta\right)-\frac{F}{r^2\sin\theta}\right)$$

Simplify the second and final terms on the right side, and then divide by $\sin\theta$ to get:

$$\frac{\partial F}{\partial t}=\nu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right)-\frac{2F}{r^2}\right).$$

c) For $t\to\infty$, the flow should be steady ($\partial F/\partial t=0$), so the result of part b) implies: $\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right)-2F=0$ for $r\neq 0$. This is an equi-dimensional equation, so it has power-law solutions. Thus, substitute $F=Ar^n$ to this equation to find $n(n+1)-2=n^2+n-2=(n+2)(n-1)=0$, which has solutions: $n=+1,-2$. The second boundary condition, $F\to 0$ as $r\to\infty$, requires the choice of $n=-2$. The first boundary condition, $F=1$ at $r=a$, determines $A=a^2$. Thus for long-time steady rotation of the sphere, $F=a^2/r^2$ and $u_\varphi=\Omega a\left(a^2/r^2\right)\sin\theta$.

d) The viscous shear stress is,

$$\sigma_{\phi r}=2\mu\frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right)=-\frac{3\mu\Omega a^3\sin\theta}{r^3},\ \text{so}\ \left[\sigma_{\phi r}\right]_{r=a}=-3\mu\Omega\sin\theta.$$

So,

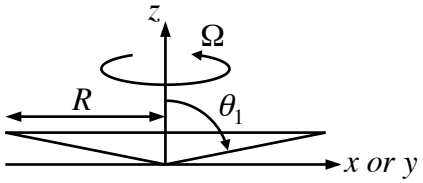
$$\text{Torque on sphere}=\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}-3\mu\Omega\sin\theta\cdot a\sin\theta\cdot a^2\sin\theta\,d\theta\,d\varphi,$$

where the integrand factors are written (stress)(moment arm)(area element). When evaluated this produces:

$$\text{Torque on sphere}=-6\pi\mu\Omega a^3\int_{\theta=0}^{\pi}\sin^3\theta\,d\theta=-8\pi\mu\Omega a^3,$$

where the minus sign indicates that the torque opposes the rotary motion, which is in the positive $\varphi$ direction.

**Exercise 9.49**. Consider the geometry of a cone and plate rheometer. A flat cone with radius $R$ and apex angle of $\theta_1$, slightly less than $\pi/2$, touches a large stationary horizontal flat plate at the origin and rotates at a constant rate $\Omega$ about the vertical $z$-axis. A viscous fluid with density $\rho$ and viscosity $\mu$ fills the gap between the cone and the plate.




a) Assuming the fluid's velocity is steady with a single component: $\mathbf{u} = \mathbf{e}_\varphi u_\varphi(r,\theta)$, use the continuity equation and the azimuthal momentum equation for creeping flow in spherical coordinates, where $r = \sqrt{x^2+y^2+z^2}$ , to find:

$$u_\varphi(r,\theta) = \Omega r \sin\theta_1 \left[ \frac{\frac{1}{2}\sin\theta \cdot \ln\left(\frac{1+\cos\theta}{1-\cos\theta}\right) + \cot\theta}{\frac{1}{2}\sin\theta_1 \cdot \ln\left(\frac{1+\cos\theta_1}{1-\cos\theta_1}\right) + \cot\theta_1} \right].$$

The boundary conditions here are $u_\varphi(r, \pi/2) = 0$, and $u_\varphi(r,\theta_1) = r\Omega\sin\theta_1$.

b) Use this to determine the polar-azimuthal shear stress: $\tau_{\theta\varphi} = \mu\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{u_\varphi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}\right]$.

c) Simplify the velocity field and shear stress results when $\theta$ and $\theta_1$ both approach $\pi/2$.

d) A torque of 3.0 N-m causes the cone to rotate with an angular velocity of 1.5 rad/s. If the radius of the cone is $R = 6.35$ cm and $90° - \theta_1 = 0.30°$, what is the viscosity of the fluid?

e) For the conditions in part d) with $\rho = 10^3$ kg/m$^3$, compare $\rho\Omega^2R^2$ to $\tau_{\theta\varphi}$. Is neglect of fluid inertia justified here?

**Solution 9.49**. a) The continuity and azimuthal momentum equation in spherical coordinates are:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(u_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(u_\varphi\right) = 0 \ , \text{ and}$$

$$\frac{\partial u_\varphi}{\partial t} + u_r\frac{\partial u_\varphi}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\varphi}{\partial\theta} + \frac{u_\varphi}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi} + \frac{1}{r}\left(u_r u_\varphi + u_\theta u_\varphi\cot\theta\right) = -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\varphi} +$$

$$\nu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_\varphi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\varphi}{\partial\varphi^2} - \frac{u_\varphi}{r^2\sin^2\theta} + \frac{2}{r^2\sin^2\theta}\frac{\partial u_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\varphi}\right).$$

For a steady single component velocity field, : $\mathbf{u} = \mathbf{e}_\varphi u_\varphi(r,\theta)$, these simplify to:

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(u_\varphi\right) = 0 \ , \text{ and}$$

$$\frac{u_\varphi}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi} = -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\varphi} + \nu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_\varphi}{\partial\theta}\right) - \frac{u_\varphi}{r^2\sin^2\theta}\right).$$

The reduced continuity equation is satisfied identically by form of the given solution. The momentum equation can be simplified to creeping flow by dropping the term on the left. In addition, since velocity field is independent of $\varphi$, assume that the pressure field is similarly independent of $\varphi$ so the first term on the right of the reduced azimuthal momentum equation is also zero. This leaves:

$$0 = 0 + \nu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_\varphi}{\partial\theta}\right) - \frac{u_\varphi}{r^2\sin^2\theta}\right)$$

subject to boundary condition: $u_\varphi(r,\pi/2) = 0$ and $u_\varphi(r,\theta_1) = r\Omega\sin\theta_1$.

The equation is linear, so assume a separation of variables solution, $u_\varphi(r,\theta) = R(r)\Theta(\theta)$ , and multiply through by $r^2$. Substituting the trial solution into the twice reduced azimuthal momentum equation produces:

$$\left[\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)\right] + \left[\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\right] = 0 \ .$$

Following the separation-of-variable solution approach, set the first set of [,]-braces equal to $\beta$, and the second set equal to $-\beta$. The ordinary differential equation that emerges for the $r$-direction is:

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \beta R \quad \text{which has solutions } R = Ar^\gamma \text{ when } \beta = \gamma(\gamma+1).$$

To satisfy the second boundary condition, $\gamma$ must be 1, so $\beta = 2$. Change the variable in the angular equation to $\eta = \cos\theta$, so that $d/d\theta = (d\eta/d\theta)(d/d\eta) = (-\sin\theta)(d/d\eta)$. This leads to:

$$\frac{1}{\Theta}\left(-\frac{\partial}{\partial\eta}\right)\left((\eta^2-1)\frac{\partial\Theta}{\partial\eta}\right) = \frac{1}{1-\eta^2} - \beta \quad \text{or} \quad \frac{\partial}{\partial\eta}\left((1-\eta^2)\frac{\partial\Theta}{\partial\eta}\right) = \left(\frac{1}{1-\eta^2} - \gamma(\gamma+1)\right)\Theta \ .$$

This is the Legendre equation degree $\gamma$ and order 1. It has solutions $P_\gamma^1(\eta)$ and $Q_\gamma^1(\eta)$ .

Here $\gamma = 1$, so

$$P_1^1(\eta) = \left(1-\eta^2\right)^{1/2} \quad \text{and} \quad Q_1^1(\eta) = \left(1-\eta^2\right)^{1/2}\left[\frac{1}{2}\ln\left(\frac{1+\eta}{1-\eta}\right) + \frac{\eta}{1-\eta^2}\right]$$

are the solutions we need. The overall solution is:

$$u_\varphi(r,\theta) = ArP_1^1(\cos\theta) + BrQ_1^1(\cos\theta) \ ,$$

where $A$ and $B$ are constants. The first boundary condition, $u_\varphi(r,\theta = \pi/2) = 0$, leads to $A = 0$ since $\cos\theta = 0$ at $\theta = \pi/2$ and $P_1^1(0) = 1$ while $Q_1^1(0) = 0$ . With $A = 0$, the second boundary condition, $u_\varphi(r,\theta_1) = r\Omega\sin\theta_1$, leads to:

$$\Omega r\sin\theta_1 = BrQ_1^1(\cos\theta_1) \quad \text{or} \quad B = \frac{\Omega\sin\theta_1}{Q_1^1(\cos\theta_1)} \ .$$

Putting all this together leads to:

$$u_\varphi(r,\theta) = \Omega r\sin\theta_1\left[\frac{\frac{1}{2}\sin\theta\cdot\ln\left(\frac{1+\cos\theta}{1-\cos\theta}\right) + \cot\theta}{\frac{1}{2}\sin\theta_1\cdot\ln\left(\frac{1+\cos\theta_1}{1-\cos\theta_1}\right) + \cot\theta_1}\right],$$

which is the desired outcome for the azimuthal velocity.

b) The approach here is to take the part a) velocity field and plug it into the shear stress formula. To simplify the effort set $C(\theta_1) = \Omega \sin\theta_1 \Big/ \left( \frac{\sin\theta_1}{2} \ln\left( \frac{1+\cos\theta_1}{1-\cos\theta_1} \right) + \cot\theta_1 \right)$, so that $u_\varphi(r,\theta) = rC(\theta_1)\left( \frac{\sin\theta}{2} \ln\left( \frac{1+\cos\theta}{1-\cos\theta} \right) + \cot\theta \right)$. Continue to the differentiations: $\partial u_\varphi / \partial \varphi = 0$, and

$$\frac{\partial}{\partial\theta}\left(\frac{u_\varphi}{\sin\theta}\right) = rC(\theta_1)\frac{\partial}{\partial\theta}\left(\frac{1}{2}\ln(1+\cos\theta) - \frac{1}{2}\ln(1-\cos\theta) + \frac{\cos\theta}{\sin^2\theta}\right)$$

$$\frac{\partial}{\partial\theta}\left(\frac{u_\varphi}{\sin\theta}\right) = rC(\theta_1)\left(\frac{-\sin\theta}{2(1+\cos\theta)} - \frac{\sin\theta}{2(1-\cos\theta)} - \frac{\sin\theta}{\sin^2\theta} - 2\frac{\cos^2\theta}{\sin^3\theta}\right)$$

$$\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{u_\varphi}{\sin\theta}\right) = C(\theta_1)\left(\frac{-\sin^2\theta}{2(1+\cos\theta)} - \frac{\sin^2\theta}{2(1-\cos\theta)} - 1 - 2\frac{\cos^2\theta}{\sin^2\theta}\right)$$

Use some trigonometric identities for simplification.

$$\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{u_\varphi}{\sin\theta}\right) = C(\theta_1)\left(\frac{-(1-\cos\theta)}{2} - \frac{1+\cos\theta}{2} - 1 - 2\frac{\cos^2\theta}{\sin^2\theta}\right) = -2C(\theta_1)\left(1 + \frac{\cos^2\theta}{\sin^2\theta}\right) = \frac{-2C(\theta_1)}{\sin^2\theta}$$

Thus, $\tau_{\theta\varphi} = \mu\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{u_\varphi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}\right] = -\mu\frac{2C(\theta_1)}{\sin^2\theta}$, which does not depend on *r*!

c) As $\theta \to \pi/2$, $\cos(\theta) \to \pi/2 - \theta$, and $\sin(\theta) \to 1$, thus

$$C(\theta_1) \cong \Omega \Big/ \left( \frac{1}{2}\ln\left( \frac{1+\pi/2-\theta_1}{1-\pi/2+\theta_1} \right) + \frac{\pi/2-\theta_1}{1} \right) \cong \Omega \Big/ \left[ 2(\pi/2-\theta_1) \right],$$

$$u_\varphi(r,\theta) \cong 2rC(\theta_1)(\pi/2-\theta) = \Omega r\left(\frac{\pi/2-\theta}{\pi/2-\theta_1}\right), \text{ and } \tau_{\theta\varphi} = -\frac{\mu\Omega}{\pi/2-\theta_1}$$

d) The torque necessary to turn the cone will be: $M = -\int_{r=0}^{R} \tau r^2 d\theta dr = 2\pi\tau R^3/3 = \frac{2\pi\mu\Omega R^3}{3(\pi/2-\theta_1)}$.

Solve this for the viscosity and evaluate:

$$\mu = \frac{3(\pi/2-\theta_1)M}{2\pi\Omega R^3} = \frac{3(0.30°\pi/180°)(3.0Nm)}{2\pi(1.50rad/s)(0.0635m)^3} = 19.5 \text{ kgm}^{-1}\text{s}^{-1}$$

e) Here: $\rho\Omega^2R^2 = (10^3\text{kg/m}^3)(1.50\text{rad/s})^2(0.0635\text{m})^2 = 9.07$ Pa, while

$\tau_{\theta\varphi} = \frac{\mu\Omega}{\pi/2-\theta_1} = \frac{(19.5Pa\cdot s)(1.50rad/s)}{0.30°(\pi/180°)} = 5.59\times10^3$ Pa, is more than 500 times higher. Thus, neglect of fluid inertia is a good approximation here.